Central elements and the Gaeta topos: An algebraic and functorial overview on coextensive varieties

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Abstract

We use the theory of central elements to establish a criterion for fp-coextensive varieties that allows to decide whether the Gaeta topos classifies indecomposable objects in terms of the indecomposability of the free algebra on one generator.

In [3] and [5] Lawvere and Schanuel introduced extensive categories as categories C with finite coproducts and pullbacks in which the canonical functor $+ : \mathsf{C}/X \times \mathsf{C}/X \to \mathsf{C}/(X+Y)$ is an equivalence. Later, in [1] an equivalent and more intuitive description was provided, namely: C is extensive if and only if it has pullbacks along injections and (finite) coproducts are disjoint and universal [1]. Such a description allowed to visualize more clearly that extensivity is a property typical of categories of "spaces" (e.g., sets, (pre)toposes, topological spaces, compact Hausdorff spaces, Stone spaces, etc.). A category is called *coextensive* if its opposite is extensive. In particular, if $\mathcal{V}$ is a variety (of universal algebras) then we say that $\mathcal{V}$ is coextensive if as an algebraic category it is coextensive. Coextensive varieties are of interest because according to [2] and more recently [4], they could bring an appropriate setting to develop algebraic geometry. Classical examples of coextensive varieties are commutative rings with unit and bounded distributive lattices. Nevertheless, there are a lot of algebraic varieties associated to non-classical logics (as well as classic logic) which are coextensive. This is the case of Boolean Algebras, Heyting algebras, MV-algebras, Gödel algebras and commutative and integral residuated lattices, to name a few. It is also worth mentioning that all these varieties are *varieties with* $\vec{0}$ *and* $\vec{1}$ (i.e. varieties in which there is a positive number N and 0-ary terms $0_1, \dots, 0_N, 1_1, \dots, 1_N$ such that $\mathcal{V} \models \bigwedge_{i=0}^{N} 0_i \approx 1_i \Longrightarrow x \approx y$) and that in all these cases, the proof of its coextensivity relies on a suitable description of those elements which concentrate all the information about direct product decompositions, namely the *central elements*. We recall [6] that given a variety $\mathcal{V}$ with $\vec{0}$ and $\vec{1}$ and $\mathbf{A} \in \mathcal{V}$, then $\vec{e} = (e_1, ..., e_N) \in A^N$ is a *central element* of $\mathbf{A}$ if there exists an isomorphism $\tau : \mathbf{A} \to \mathbf{A}_1 \times \mathbf{A}_2$, such that $\tau(e_i) = (0_i^{\mathbf{A}_1}, 1_i^{\mathbf{A}_2})$ for every $1 \le i \le N$.

If C is a small extensive category, the *Gaeta Topology* on C is the (Grothendieck) topology $J_{\mathcal{G}}$ generated by all finite families $X_i \to X$, such that $\Sigma X_i \to X$ is an isomorphism. The *Gaeta topos* $\mathcal{G}(\mathsf{C})$, is the topos of sheaves on the site $(\mathsf{C}, J_{\mathcal{G}})$. In concrete examples ([7], [4]), it has been proved that the Gaeta topos is the classifying topos of the theory of "connected" objects, which can be considered as the ones who does not admit non-trivial binary coproduct decompositions. Naturally, when considering coextensive categories, the Gaeta topology and the Gaeta topos are related with decompositions into finite products and "indecomposable" (by

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direct-products) objects. That is to say, indecomposable objects are precisely the connected objects in the opposite category. However, in the practice it seems not quite easy to provide an axiomatization of the theory of indecomposable objects when regarding varieties in a more general setting. Fortunately, with the aid of the theory of central elements [6] this obstacle can be tackled. In this context one may be tempted to study those coextensive varieties such that the Gaeta topos classifies the theory of indecomposable objects. As a first step we restrict the problem to those coextensive varieties $\mathcal{V}$ such that the full subcategory of finitely presented algebras of $\mathcal{V}$, $\mathsf{Mod}_{\mathsf{fp}}(\mathcal{V})$ is coextensive. We call such varieties *fp-coextensive*. Moreover, if $\mathcal{V}$ is fp-coextensive, we write $\mathcal{G}(\mathcal{V})$ for the Gaeta topos determined by the extensive category $\mathsf{Mod}_{\mathsf{fp}}(\mathcal{V})^{\mathsf{op}}$.

By using the characterization of coextensive varieties presented in [8], in this talk we provide an algebraic description of those fp-coextensive varieties $\mathcal{V}$ such that $\mathcal{G}(\mathcal{V})$ classifies the theory of $\mathcal{V}$-indecomposable objects by means of the behavior of the free algebra of $\mathcal{V}$ on one generator $\mathbf{F}_{\mathcal{V}}(x)$. Concretely, we will prove the following:

Theorem 1. Let $\mathcal{V}$ be a fp-coextensive variety. Then, the following are equivalent:

(1) $\mathcal{G}(\mathcal{V})$ is a classifying topos for $\mathcal{V}$-indecomposable objects.

(2) $\mathbf{F}_{\mathcal{V}}(x)$ is indecomposable in Set.

Afterwards, we apply this result to decide whether the Gaeta topos classifies indecomposable objects in some particular classes of algebras associated to non-classical logics, namely Bounded distributive lattices, Heyting algebras, Gödel algebras and MV-algebras. Some of these results are known and some others are new.

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