The Monotone-Light Factorization for 2-categories via 2-preorders

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It is shown that the reflection $2Cat \to 2Preord$ of the category of all 2-categories into the category of 2-preorders determines a monotone-light factorization system on $2Cat$ and that the light morphisms are precisely the 2-functors faithful on 2-cells with respect to the vertical structure. In order to achieve such result it was also proved that the reflection $2Cat \to 2Preord$ has stable units, a stronger condition than admissibility in categorical Galois theory, and that the 2-functors surjective both on horizontally and on vertically composable triples of 2-cells are effective descent morphisms in $2Cat$.

Every map $\alpha : A \to B$ of compact Hausdorff spaces has a factorization $\alpha = me$ such that $m : C \to B$ has totally disconnected fibres and $e : A \to C$ has only connected ones. This is known as the classical monotone-light factorization of S. Eilenberg [3] and G. T. Whyburn [7].

Consider now, for an arbitrary functor $\alpha : A \to B$, the factorization $\alpha = me$ such that m is a faithful functor and e is a full functor bijective on objects. This familiar factorization for categories is as well monotone-light. Meaning that both factorizations are special and very similar cases of categorical monotone-light factorization in an abstract category $\mathbb{C}$, with respect to a full reflective subcategory $\mathbb{X}$, as was studied in [1]. What we shall show is that there is also a monotone-light factorization for 2-categories, very similar to the one given before for categories if one ignores the horizontal composition of 2-cells.

It is well known that any full reflective subcategory $\mathbb{X}$ of a category $\mathbb{C}$ gives rise, under mild conditions, to a factorization system $(\mathcal{E}, \mathcal{M})$. Hence, each of the three reflections $CompHaus \to Prof$, of compact Hausdorff spaces into Stone(profinite) spaces, $Cat \to Preord$, of categories into preorders, and now $2Cat \to 2Preord$, of 2-categories into 2-preorders yields its own reflective factorization system.

Moreover, the process of simultaneously stabilizing $\mathcal{E}$ and localizing $\mathcal{M}$, in the sense of [1], was already known to produce a new non reflective and stable factorization system $(\mathcal{E}^{'}, \mathcal{M}^{*})$ for the adjunctions $CompHaus \to Prof$ and $Cat \to Preord$. Which is just the (**Monotone**, **Light**)-factorization mentioned above. But this process does not work in general, being the monotone-light factorizations for the reflections $CompHaus \to Prof$ and $Cat \to Preord$ two rare examples. Nevertheless, we shall prove that the (**Full on 2-Cells and Bijective on Objects and Morphisms**, **Faithful on 2-Cells**)-factorization (notice that "full" and "faithful" here are with respect to the vertical composition) for 2-categories is another instance of a successful simultaneous stabilization and localization.

What guarantees the success is the following pair of conditions, which hold in the three cases:

1. the reflection $I : \mathbb{C} \to \mathbb{X}$ has stable units (in the sense of [2]);

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2. for each object B in $\mathbb{C}$, there is a monadic extension (it is said that (E,p) is a monadic extension of B, or that p is an effective descent morphism, if the pullback functor $p^* : \mathbb{C}/B \to \mathbb{C}/E$ is monadic) (E,p) of B such that E is in the full subcategory $\mathbb{X}$.

Indeed, the two conditions (1) and (2) trivially imply that the $(\mathcal{E},\mathcal{M})$-factorization is locally stable, which is a necessary and sufficient condition for $(\mathcal{E}^{'},\mathcal{M}^*)$ to be a factorization system (see the central result of [1]).

The three reflections may be considered as admissible Galois structures (in which all morphisms are considered), in the sense of categorical Galois theory, since having stable units implies admissibility. Therefore, in the three cases, for every object B in $\mathbb{C}$, we know that the full subcategory $TrivCov(B)$ of $\mathbb{C}/B$, determined by the trivial coverings of B (i.e., the morphisms over B in $\mathcal{M}$), is equivalent to $\mathbb{X}/I(B)$. Moreover, the categorical form of the fundamental theorem of Galois theory gives us even more information on each $\mathbb{C}/B$ using the subcategory $\mathbb{X}$. It states that the full subcategory $Spl(E,p)$ of $\mathbb{C}/B$, determined by the morphisms split by the monadic extension (E,p) of B, is equivalent to the category $\mathbb{X}^{Gal(E,p)}$ of internal actions of the Galois pregroupoid of (E,p).

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