Sahlqvist correspondence for deductive systems

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In this talk we present a Sahlqvist Correspondence Theorem [11] for finitary protoalgebraic logics. Our proof is based on an extension of Sahlqvist theory to various fragments of IPC. A formula in the language

$$\mathcal{L} \quad ::= \quad x \mid \varphi \wedge \psi \mid \varphi \vee \psi \mid \varphi \to \psi \mid \neg\varphi \mid 0 \mid 1$$

is said to be

(i) a *Sahlqvist antecedent* if it is constructed from variables, negative formulas, and the constants 0 and 1 using only $\wedge$ and $\vee$;

(ii) a *Sahlqvist implication* if either it is positive, or it has the form $\neg\varphi$ for a Sahlqvist antecedent φ, or it has the form $\varphi \to \psi$ for a Sahlqvist antecedent φ and a positive formula ψ.

Lastly, a *Sahlqvist quasiequation* is a universal sentence of the form

$$\forall \vec{x}, y, z((\varphi_1(\vec{x}) \wedge y \leqslant z \,\&\, \ldots \,\&\, \varphi_n(\vec{x}) \wedge y \leqslant z) \Longrightarrow y \leqslant z),$$

where y, z are distinct variables that do not occur in $\varphi_1, \ldots, \varphi_n$ and each φ_i is constructed from Sahlqvist implications using only $\wedge$ and $\vee$.

Remark 1. The focus on quasiequations (as opposed to formulas or equations) is necessary as we deal with fragments where equations have a very limited expressive power. ⊠

Let PSL, (b)ISL, PDL, IL, and HA be, respectively, the varieties of pseudocomplemented semilattices, (bounded) implicative semilattices, pseudocomplemented distributive lattices, implicative lattices, and Heyting algebras. Furthermore, given a poset $\mathbb{X}$, let $\mathsf{Up}(\mathbb{X})$ be the Heyting algebra of its upsets.

Theorem 2. *The following holds for every variety* K *between* PSL, (b)ISL, PDL, IL, *and* HA *and every Sahlqvist quasiequation* Φ *in the language of* K:

(i) Canonicity*: For every* $\boldsymbol{A} \in \mathsf{K}$*, if* $\boldsymbol{A}$ *validates* Φ*, then also* $\mathsf{Up}(\boldsymbol{A}_*)$ *validates* Φ*, where* $\boldsymbol{A}_*$ *is the poset of the meet irreducible filters of* $\boldsymbol{A}$*;*

(ii) Correspondence*: There exists an effectively computable sentence* $\mathsf{fo}(\Phi)$ *in the language of posets such that* $\mathsf{Up}(\mathbb{X}) \vDash \Phi$ *iff* $\mathbb{X} \vDash \mathsf{fo}(\Phi)$*, for every poset* $\mathbb{X}$*.*

To prove Theorem 2, first we extend Sahlqvist Theorem to IPC using Gödel translation of IPC into S4 [7] and its duality theoretic interpretation (see, e.g., [3]). Then, we develop a discrete duality for each variety K as above (cf. [1]) and utilize it to extend Sahlqvist Theorem to the corresponding fragment of IPC.

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A *logic* $\vdash$ is a finitary substitution invariant consequence relation on the set of formulas of some language. Let $\vdash$ be a logic and $\boldsymbol{A}$ an algebra. A subset F of $\boldsymbol{A}$ is said to be a *deductive filter* of $\vdash$ on $\boldsymbol{A}$ if it is closed under the interpretation of the rules valid in $\vdash$. When ordered under the inclusion relation, the set of deductive filters of $\vdash$ on $\boldsymbol{A}$ forms an algebraic lattice $\mathsf{Fi}_{\vdash}(\boldsymbol{A})$ with semilattice of compact elements $\mathsf{Fi}^{\omega}_{\vdash}(\boldsymbol{A})$. Lastly, the poset of meet irreducible elements of $\mathsf{Fi}_{\vdash}(\boldsymbol{A})$ will be denoted by $\mathsf{Spec}_{\vdash}(\boldsymbol{A})$.

In order to extend Sahlqvist Correspondence to arbitrary logics, recall that a logic $\vdash$ is said to have

(i) The *inconsistency lemma* (IL) [10] if for every $n \in \mathbb{Z}^+$ there is a finite set of formulas $\sim_n(x_1, \dots, x_n)$ such that for every set of formulas $\Gamma \cup \{\varphi_1, \dots, \varphi_n\}$,

$$\Gamma \cup \{\varphi_1, \dots, \varphi_n\} \text{ is inconsistent } \text{ iff } \Gamma \vdash \sim_n(\varphi_1, \dots, \varphi_n);$$

(ii) The *deduction theorem* (DT) [2] if for every $n, m \in \mathbb{Z}^+$ there is a finite set $(x_1, \dots, x_n) \Rightarrow_{nm} (y_1, \dots, y_m)$[1] of formulas such that for every set of formulas $\Gamma \cup \{\psi_1, \dots, \psi_n, \varphi_1, \dots, \varphi_m\}$,

$$\Gamma, \psi_1, \dots, \psi_n \vdash \varphi_1, \dots, \varphi_m \text{ iff } \Gamma \vdash (\psi_1, \dots, \psi_n) \Rightarrow_{nm} (\varphi_1, \dots, \varphi_m);$$

(iii) The *proof by cases* (PC) [4, 5] if for every $n, m \in \mathbb{Z}^+$ there is a finite set of formulas $(x_1, \dots, x_n) \curlyvee_{nm} (y_1, \dots, y_m)$ such that for every set of formulas $\Gamma \cup \{\psi_1, \dots, \psi_n, \varphi_1, \dots, \varphi_m, \gamma\}$,

$$\Gamma, \psi_1, \dots, \psi_n \vdash \gamma \text{ and } \Gamma, \varphi_1, \dots, \varphi_m \vdash \gamma \text{ iff } \Gamma, (\psi_1, \dots, \psi_n) \bigcurlyvee_{nm} (\varphi_1, \dots, \varphi_m) \vdash \gamma.$$

A formula φ in $\mathcal{L}$ is *compatible* with a logic $\vdash$ when

(i) If 0 (resp. 1) occurs in φ, then $\vdash$ has the IL (resp. the IL or the DT);

(ii) If $\neg$ (resp. $\rightarrow, \vee$) occurs in φ, then $\vdash$ has the IL (resp. DT, PC).

In this case, for every $k \in \mathbb{Z}^+$ we associate a finite set $\boldsymbol{\varphi}^k(\vec{x}_1, \dots, \vec{x}_n)$ of formulas of $\vdash$ (where each $\vec{x}_i$ is a sequence of length k) with φ as follows:

(i) If $\varphi = x_i$, then $\boldsymbol{\varphi}^k := \{\vec{x}_i\}$;

(ii) If $\varphi = \psi \wedge \gamma$, then $\boldsymbol{\varphi}^k := \boldsymbol{\psi}^k \cup \boldsymbol{\gamma}^k$;

(iii) If $\varphi = \neg\psi$, then $\vdash$ has the IL and, therefore, we set $\boldsymbol{\varphi}^k := \sim_m(\gamma_1, \dots, \gamma_m)$ where $\boldsymbol{\psi}^k = \{\gamma_1, \dots, \gamma_m\}$;

(iv) The cases where φ has the form $\psi \rightarrow \gamma$ or $\psi \vee \gamma$ are handled similarly to the previous one.

By a *Sahlqvist quasiequation for a logic* $\vdash$ we signify a Sahlqvist quasiequation

$$\Phi = \forall \vec{x}, y, z((\varphi_1(x_1, \dots, x_m) \wedge y \leqslant z \,\&\, \dots \,\&\, \varphi_n(x_1, \dots, x_m) \wedge y \leqslant z) \Longrightarrow y \leqslant z),$$

where $\varphi_1, \dots, \varphi_n$ are compatible with $\vdash$. With it, we associate the set $\mathcal{R}(\Phi)$ of metarules for $\vdash$ of the form

$$\frac{\Gamma, \boldsymbol{\varphi}_1^k(\vec{\gamma}_1, \dots, \vec{\gamma}_m) \vdash \psi, \dots, \Gamma, \boldsymbol{\varphi}_n^k(\vec{\gamma}_1, \dots, \vec{\gamma}_m) \vdash \psi}{\Gamma \vdash \psi.}$$

where $k \in \mathbb{Z}^+$, $\Gamma \cup \{\psi\}$ is a set of formulas, and $\vec{\gamma}_1, \dots, \vec{\gamma}_m$ are sequences of formulas of length k.

A logic is *protoalgebraic* if there exists a set of formulas $\Delta(x, y)$ such that $\emptyset \vdash \Delta(x, x)$ and $x, \Delta(x, y) \vdash y$. Our general Sahlqvist Correspondence Theorem takes the following form:

[1] We signify that $\Rightarrow_{nm}$ is a set of formulas in the variables $x_1, \dots, x_n, y_1, \dots, y_m$ by the more suggestive notation $(x_1, \dots, x_n) \Rightarrow_{nm} (y_1, \dots, y_m)$. A similar convention applies to Condition (iii).

Sahlqvist Correspondence. *Let Φ be a Sahlqvist quasiequation for a protoalgebraic logic $\vdash$. Then,*

$$\vdash \textit{ validates the metarules in } \mathcal{R}(\Phi) \quad \textit{iff} \quad \mathsf{Spec}_{\vdash}(\boldsymbol{A}) \vDash \mathsf{fo}(\Phi) \textit{ for every algebra } \boldsymbol{A}.$$

As a consequence, we obtain for instance that a protoalgebraic logic with an IL satisfies a generalization of the excluded middle law (resp. of the bounded top width n formula) iff it is semisimple (resp. principal upsets in $\mathsf{Spec}_{\vdash}(\boldsymbol{A})$ have at most n maximal elements, for every algebra $\boldsymbol{A}$) [8, 9]. The results of this talk are collected in [6].

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