Abstracting sheafification as a tripos-to-topos adjunction

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In topos theory, localic toposes constitute an important class of examples of Grothendieck's toposes of sheaves. A localic topos is defined as the category $\mathsf{Shv}(A)$ of shaves on a locale A. Recall that sheaves are usually described as *presheaves on A satisfying a glueing condition*, and it is well-known that the category of sheaves $\mathsf{Shv}(A)$ and that of presheaves $\mathsf{PShv}(A)$ are connected via the so-called *sheafification* functor, i.e. the left adjoint functor s to the inclusion

$$\begin{tikzcd} \mathsf{PShv}(A) \arrow[r, bend left, "s"] \arrow[r, phantom, "\perp"] & \mathsf{Shv}(A). \arrow[l, bend left, "i"] \end{tikzcd}$$

Motivated by the construction of the category of sheaves on a locale, Hyland, Johnstone and Pitts introduced in [2] a generalization of this construction called *tripos-to-topos* construction. The idea is that given a tripos $P\colon \mathsf{Set}^{\mathrm{op}} \longrightarrow \mathsf{Pos}$, acronym of *topos-representing indexed pre-orders sets*, we can construct a topos T_P that generalises the notion of topos of sheaves on a locale. This is a proper generalization since each tripos-to-topos construction produces a Lawvere-Tierney elementary topos, but not necessarily a localic topos or more generally a Grothendieck topos. A remarkable example of such non-sheaf toposes are realizability toposes.

The main purpose of our work is to show that the sheafification adjunction happens to be the instance of a more abstract adjunction between tripos-to-topos constructions induced from an adjunction between two triposes in the sense of [2] satisfying suitable conditions. In detail:

Theorem. *Let $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathsf{InfSl}$ be a tripos such that*

1. *$\mathcal{C}$ has weak dependent products;*
2. *the predicate classifier Ω has a power object $\mathrm{P}\Omega$ in $\mathcal{C}$;*
3. *$\mathcal{C}$ admits a proper factorization system $\langle \mathcal{E}, \mathcal{M}\rangle$, such that every epi of $\mathcal{E}$ splits.*

We associate to P a tripos $P^{\exists}\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathsf{InfSl}$ and its tripos-to-topos construction $\mathsf{T}_{P^{\exists}}$, called ***presheaf topos substitute****, for which*

1. *the tripos-to-topos construction T_P is included in $\mathsf{T}_{P^{\exists}}$ via $i\colon \mathsf{T}_P \longrightarrow \mathsf{T}_{P^{\exists}}$;*
2. *the inclusion $i\colon \mathsf{T}_P \longrightarrow \mathsf{T}_{P^{\exists}}$ has a left adjoint*

$$\begin{tikzcd} \mathsf{T}_{P^{\exists}} \arrow[r, bend left, "s"] \arrow[r, phantom, "\perp"] & \mathsf{T}_P. \arrow[l, bend left, "i"] \end{tikzcd}$$

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As corollaries, we obtain the following (where Set is the category of sets and functions definable in the classical set theory ZFC):

Corollary. *For every tripos* $P\colon \mathsf{Set}^{\mathrm{op}} \longrightarrow \mathsf{InfSl}$ *on* Set *we have an adjunction of toposes*

$$\begin{tikzcd} \mathsf{T}_{P^{\exists}} \arrow[r, bend left, "s"] \arrow[r, phantom, "\perp"] & \mathsf{T}_{P}. \arrow[l, bend left, "i"] \end{tikzcd}$$

Corollary. *Let* $A^{(-)}\colon \mathsf{Set}^{\mathrm{op}} \longrightarrow \mathsf{InfSl}$ *be the localic tripos. Then the category* $\mathsf{T}_{A^{\exists}}$ *is precisely the topos of presheaves* $\mathsf{PShv}(A)$*, and the arrow* $s\colon \mathsf{PShv}(A) \longrightarrow \mathsf{Shv}(A)$ *is precisely the sheafification.*

The key tools to prove our results is the characterization of the full existential completion and its tripos-to-topos presented in [5], the notion of quotient completion in [4] and adiunctions between them in [3]: given a tripos $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathsf{InfSl}$, the doctrine $P^{\exists}\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathsf{InfSl}$ is defined as the full existential completion of the tripos P . Under the assumptions of our previous theorem, we show that $P_{\exists}$ is a tripos (and then $\mathsf{T}_{P^{\exists}}$ is a topos). The adjunction between $\mathsf{T}_{P^{\exists}}$ and T_{P} follows from the universal properties of the full existential completion and elementary quotient completions [3]. From a result presented in [5], since $P^{\exists}$ is the full existential completion of P, we conclude that $\mathsf{T}_{P^{\exists}} \equiv (\mathcal{G}_P)_{\mathsf{ex/lex}}$ where $\mathcal{G}_P$ is the Grothendieck category of P.

In the localic case, this result together with the well-known equivalence $\mathsf{PShv}(A) \equiv (A_+)_{\mathsf{ex/lex}}$, see [1, 6], and the observation that $A_+ \equiv \mathcal{G}_A$ where $\mathcal{G}_A$ is the Grothendieck category of the localic tripos $A^{(-)}\colon \mathsf{Set}^{\mathrm{op}} \longrightarrow \mathsf{InfSl}$, see [5], allows us to conclude that $\mathsf{T}_{A^{\exists}} \equiv \mathsf{PShv}(A)$ and that the adjunction we obtain is precisely the sheafification one.

A remarkable non-localic example of doctrinal sheaves is the effective topos $\mathsf{Eff} \equiv \mathsf{T}_R$, where R is the realizability tripos, whose doctrinal presheaves is $\mathsf{T}_{R^{\exists}}$. We leave to future work to investigate whether $\mathsf{T}_{R^{\exists}}$ is itself a realizability topos, too.

References

[1] A. Carboni. Some free constructions in realizability and proof theory. *J. Pure Appl. Algebra*, 103:117–148, 1995.

[2] J.M.E. Hyland, P.T. Johnstone, and A.M. Pitts. Tripos theory. *Math. Proc. Camb. Phil. Soc.*, 88:205–232, 1980.

[3] M.E. Maietti, F. Pasquali, and G. Rosolini. Quasi-toposes as elementary quotient completions. *arXiv*, `https://arxiv.org/abs/2111.15299`, 2021.

[4] M.E. Maietti and G. Rosolini. Elementary quotient completion. *Theory App. Categ.*, 27(17):445–463, 2013.

[5] M.E. Maietti and D. Trotta. Generalized existential completion and their regular and exact completions. *arXiv*, `https://arxiv.org/abs/2111.03850`, 2021.

[6] M. Menni. A characterization of the left exact categories whose exact completions are toposes. *Journal of Pure and Applied Algebra*, 177(3):287–301, 2003.