Non-distributive logics as evidential logics

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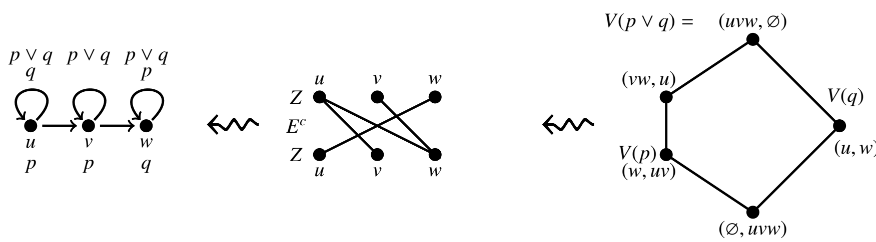
The term 'non-distributive logics'(cf. [1]) refers to the wide family of non-classical propositional logics in which the distributive laws $\alpha \wedge (\beta \vee \gamma) \vdash (\alpha \wedge \beta) \vee (\alpha \wedge \gamma)$ and $(\alpha \vee \beta) \wedge (\alpha \vee \gamma) \vdash \alpha \vee (\beta \wedge \gamma)$ do not need to be valid. Since the rise of very well known instances such as quantum logic [9], interest in non-distributive logics has been building steadily over the years. Techniques and ideas have come from pure mathematical areas such as lattice theory, duality and representation (cf. [8, 6]), and areas in mathematical logic such as algebraic proof theory (cf. [5, 2]), but also from the philosophical and formal foundations of quantum physics [7, 1], philosophical logic [11] theoretical computer science and formal linguistics [10].

In this talk, we present a type of (Kripke-style) relational semantics for non-distributive logics which is based on reflexive directed graphs (i.e. tuples (Z, E) such that Z is a set and $E \subseteq Z \times Z$ is reflexive), as in the left-hand side of the picture below. Via an intermediate structure, every such graph can be associated with a complete lattice, as in the right-side of the picture. Thanks to this fact, the interpretation of non-distributive logics on lattice-based algebras transfers to graph-based relational models. The topic we will discuss in this presentation is part of an ongoing line of research [4], whose developments are technically rooted in dual characterization results and insights from unified correspondence theory.



Interestingly, the distinguishing feature of this *graph-based semantics* is that, at any given state z of any such model, a formula ϕ can be satisfied ($z \Vdash \phi$), refuted ($z \succ \phi$), or *neither*. We will argue that, thanks to this feature, graph-based models support an interpretation of non-distributive logics as *evidential logics*, i.e. logics aimed at capturing correct reasoning in situations in which the notion of truth and falsity is based on the availability of evidence (in support or against a proposition). These notions of truth and falsity are even more refined than their intuitionistic analogues, since, in order to refute a formula, it is not enough there being lack of evidence supporting it, but rather, evidence against it needs to be presented.

In this talk we will show that a systematic relationship can be established between the first-order correspondents of all Sahlqvist modal reduction principles[1] on Kripke frames and graph-based frames. For instance, the Sahlqvist modal reduction principle $\Box p \vdash p$, which corresponds to the reflexivity condition $\Delta \subseteq R$ on Kripke frames (W, R), corresponds to the first-order condition $E \subseteq R$ on graph-based frames (Z, E, R).

More in general, the first order correspondents of Sahlqvist modal reduction principles for graph-based semantics can be formulated as the E-counterparts of their first-order correspondents on Kripke frames. This gives rise to the notion of *parametric correspondence* [3] in graph-based frames.

[1] *Sahlqvist modal reduction principles* are sequents of the form $\phi[\alpha(p)/x] \vdash \psi[\chi(p)/y]$ or $\phi[\chi(p)/x] \vdash \psi[\beta(p)/y]$, where $\phi(x)$ and $\beta(p)$ are built out of $\Diamond$ connectives, $\psi(x)$ and $\alpha(p)$ out of $\Box$ connectives, and $\chi(p)$ out of both $\Box$ and $\Diamond$ connectives.

Besides being of technical interest, this result lends itself as a base for further and more conceptual investigations on how a given interpretation of a modal axiom transfers from one semantic context to another. For instance, We show that the first order correspondents of Sahlqvist modal reduction principles on graph-based semantics can be seen as *lifted* versions of their first order correspondents on the Kripke frames under a suitable notion of lifting. On the other hand, first order correspondents on graph-based frames reduce to the first correspondents on Kripke-frames when the relation defining them, that is, the “parameter”, is identity.

When comparing the meaning of $\Box p \vdash p$ on Kripke models and on graph-based models under the epistemic understanding of $\Box$, the factivity reading of the axiom corresponds to the reflexivity condition on Kripke models requiring the agent to not exclude the true world. Similarly, the E-counterpart of reflexivity on a graph-based frame requires the agent to not exclude any world which is an E-successor of the true one, which corresponds to factivity in a setting in which different states of affairs might be *inherently indiscernible* (and their inherent indiscernibility is encoded by the relation E).

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