Filtral pretoposes and compact Hausdorff locales

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In their recent work [3] Marra and Reggio characterized the category of compact Hausdorff spaces as the unique, up to equivalence, non-trivial, well-pointed, filtral pretopos with set-indexed copowers of its terminal object. Recall that a pretopos is a category which has disjoint and universal sums, a feature typical of categories of spaces, while it has pullback-stable image factorizations and every equivalence relation has a coequalizer and is the kernel pair of it, a feature typical of algebraic categories.

In view of the significant role that compact Hausdorff locales have in the development of mathematics internally in a topos it would be interesting to know if the category of compact Hausdorff locales admits a similar "pointless" characterization. The approach adopted in [3] has the deficit, from our perspective, that it stresses the role of points right from the beginning. It has though the advantage of introducing the key concept of filtrality, which is fundamental for our approach too. Filtrality is the property of having enough objects whose lattice of subobjects is the dual of a Stone frame. Without resorting to the classically valid equivalence with the respective topological spaces, one can show that the category CHLoc of compact Hausdorff locales is a pretopos [2] and moreover it is filtral ([1], D4.6.8).

We show that, for any filtral pretopos, there is a functor to CHLoc, namely the one that assigns to an object of such a pretopos the lattice of subobjects of it with its dual order. For that we needed to show first that a closed quotient of a compact Hausdorff locale is Hausdorff, a result that may have its own independent interest. Indeed one has

Theorem 1. *If $f : Y \to X$ is a closed surjection of locales and Y is compact Hausdorff then X is compact Hausdorff (and hence the surjection is proper).*

Proof: Every compact Hausdorff locale admits a proper surjection from a Stone locale and the composite fe of f with a proper surjection e is proper iff f is proper, so assume that Y is Stone. As such it is subfit, i.e every open sublocale of it is intersection of closed ones. Being compact Hausdorff it is also normal. Hence X is also compact and normal. It suffices, that X is also subfit, because then following [4] Proposition 4.4, X is regular, hence X is compact Hausdorff. The result follows from the next proposition. □

Proposition 2. *If $f : Y \to X$ is a closed surjection of locales and Y is subfit then X is subfit.*

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Proof: If the nucleus $j = u \to -$ corresponds to an open sublocale U of X then $f^{-}j = f^{*}u \to -$ corresponds to the inverse image of U in Y. Since Y is subfit we have that $f^{*}u \to - = \bigwedge_i (v_i \vee -)$ in the frame of nuclei on OY. Hence its direct image is $f_{+}f^{-}j = f_{+}(\bigwedge_i v_i \vee -) = \bigwedge_i f_{+}(v_i \vee -)$. Each $f_{+}(v_i \vee -)$ is closed by the assumption of closedness of f. On the other hand, while for each $j \in NX$, in general $j \leq f_{+}f^{-}j$, for $j = u \to -$ we moreover have $f_{+}f^{-}j = f_{*}(f^{*}u \to f^{*}-) \leq j = u \to -$ because if $w \leq f_{*}(f^{*}u \to f^{*}v)$, then $f^{*}w \leq f^{*}u \to f^{*}v$, equivalently $f^{*}(w \wedge u) \leq f^{*}v$, so we conclude that $w \leq u \to v$ by the surjectivity of f. □

Filtrality of a pretopos $\mathcal{K}$ gives that, for each $X \in \mathcal{K}$ the lattice of its subobjects, with its dual order, is the frame of a closed (because of Frobenious reciprocity) quotient of a Stone locale, so the assignment $X \mapsto \mathrm{Sub}(X)^{op}$ is the object part of a functor from the filtral pretopos $\mathcal{K}$ to the category of compact Hausdorff locales CHLoc, with image $f[-]$ of subobjects as direct image of the locale map.

Theorem 3. *For a filtral pretopos $\mathcal{K}$, the functor $\mathrm{Sub}(-)^{op} \colon \mathcal{K} \to \mathrm{CHLoc}$ is full on subobjects, faithful, preserves (regular) epis and equalizers. Assuming further that the product $S = S_1 \times S_2$ of two filtral objects is filtral, the map $B_1 \coprod B_2 \to B$ involving the respective boolean algebras of complemented subobjects is injective and the unique map from $\mathrm{Sub}\boldsymbol{1}$ to the terminal locale (which is compact Hausdorff) is a surjection, then it preserves finite products as well.*

Proof: Concerning preservation of equalizers, upon which faithfulness also hinges, for a pair of maps $f, g : Y \to Z$ in $\mathcal{K}$ with equalizer $X \to Y$, by the description of equalizers of Hausdorff locales in [5], the equalizer of $f[-]$, $g[-]$ is given as $\downarrow(\bigwedge\{f^{-1}[S] \vee g^{-1}[\sim S] \in OX \mid S \leq Z\})$ (taking into account the existence of dual pseudo-complements in $\mathrm{Sub}(Z)$.) In the less obvious direction, $\mathrm{Sub}(X)^{op}$ is contained in the equalizer because X is below each $f^{-1}[S] \wedge g^{-1}[\sim S]$. Indeed, for each $S \leq Z$ we have $X \wedge f^{-1}[S] = X \wedge g^{-1}[S]$ (because $T \leq X \wedge f^{-1}[S]$ iff $T \leq X$ and $T \leq f^{-1}[S]$, equivalently $T \leq X$ and $f[T] \leq S$, iff $T \leq X$ and $g[T] \leq S$, or, $T \leq X$ and $T \leq g^{-1}[S]$.) Then

$$\begin{aligned} X \wedge (f^{-1}[S] \vee g^{-1}[\sim S]) &= (X \wedge f^{-1}[S]) \vee (X \wedge g^{-1}[\sim S]) \\ &= (X \wedge f^{-1}[S]) \vee (X \wedge f^{-1}[\sim S]) \\ &= X \wedge f^{-1}[S \vee \sim S]) = X \wedge Y = X \qquad \square \end{aligned}$$

If $\mathcal{K}$ has copowers of 1, and in the base topos Stone locales have enough points, one gets that the functor is also covering, thus an equivalence. The characterization of [3] is recovered this way.

References

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