Syntactic completeness of proper display calculi

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In recent years, research in structural proof theory has focused on *analytic calculi* [14, 3, 10, 2, 18, 19], understood as those calculi supporting a *robust* form of cut elimination, i.e. one which is preserved by adding rules of a specific shape (the analytic rules). Important results on analytic calculi have been obtained in the context of various proof-theoretic formalisms: (classes of) axioms have been identified for which equivalent correspondences with analytic rules have been established algorithmically or semi-algorithmically. This strand of research has been developed in the context of *sequent and labelled calculi* [16, 17, 15, 14], *sequent and hypersequent calculi* [3, 12, 13], and *(proper) display calculi* [11, 4, 10].

In [10], which is the contribution in the line of research described above to which the results discussed in the present talk most directly connect, a characterization, analogous to the one of [4],[1] of the property of being properly displayable[2] is obtained for arbitrary normal (D)LE-logics[3] via a systematic connection between proper display calculi and generalized Sahlqvist correspondence theory (aka unified correspondence [5, 6, 7, 8]). Thanks to this connection, general meta-theoretic results are established for properly displayable (D)LE-logics. In particular, in [10], the properly displayable (D)LE-logics are syntactically characterized as the logics axiomatised by *analytic inductive axioms* (namely axioms of a given syntactic shape, see [10, Definition 51 and 55]); moreover, the same algorithm ALBA which computes the first-order correspondent of (analytic) inductive (D)LE-axioms can be used to effectively compute their corresponding analytic structural rule(s). In [1], following [9], residuated families of unary and binary connectives are studied parametrically in group actions on the coordinates of the relations associated with the connectives.

The *semantic* equivalence between each analytic inductive axiom $\varphi \vdash \psi$ and its corresponding analytic structural rule(s) $R_1, \ldots, R_n$, discussed in [10], is an immediate consequence of the soundness of the rules of ALBA on perfect normal (distributive) lattice expansions. On the *syntactic* side, a description of the derivation, which relies on the proof-theoretic version of Ackermann's Lemma and therefore involves cuts, is presented in [4]. However, an effective procedure was still missing for building *cut-free* derivations of $\varphi \vdash \psi$ in the proper display calculus obtained by adding $R_1, \ldots, R_n$ to the basic proper display calculus D.LE (resp. D.DLE) of the basic normal (D)LE-logic. Such an effective procedure would establish, via syntactic means, that for any properly displayable (D)LE-logic L, the proper display calculus for L—i.e. the calculus obtained by adding the analytic structural rules obtained from the axioms of L to the basic calculus D.LE (resp. D.DLE)—derives all the theorems (or derivable sequents) of L. This is what we refer to as the *syntactic completeness* of the proper display calculus for L with respect to any analytic (D)LE-logic L. This syntactic completeness result for all properly displayable logics in arbitrary (D)LE-signatures is the main contribution of the present research. It is perhaps worth emphasizing that we do not just show that any analytic inductive axiom is derivable in its corresponding proper display calculus, but we also provide an algorithm to generate a *cut-free* derivation of a particular shape that we refer to as being *in pre-normal form*.

[1] For a comparison between the characterizations in [4] and in [10], see [10, Section 9].

[2] A display calculus is *proper* if every structural rule is closed under uniform substitution. A logic is *(properly) displayable* if it can be captured by some (proper) display calculus (see [10, Section 2.2]).

[3] Normal (D)LE-logics are those logics algebraically captured by varieties of normal (distributive) lattice expansions, i.e. (distributive) lattices endowed with additional operations that are finitely join-preserving or meet-reversing in each coordinate, or are finitely meet-preserving or join-reversing in each coordinate.

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