Internal Factorisation Systems

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In a category $\mathbb{C}$, a morphism f is *orthogonal* to a morphism g (written $f \downarrow g$) if for all morphisms u and v in $\mathbb{C}$ such that $vf = gu$, there is a unique morphism z satisfying $zf = u$ and $gz = v$. A *factorisation system* on a category $\mathbb{C}$ may then be defined as a pair, $(\mathcal{E}, \mathcal{M})$, of classes of morphisms of $\mathbb{C}$ which both contain all isomorphisms of $\mathbb{C}$ and are closed under composition, such that for all e in $\mathcal{E}$ and m in $\mathcal{M}$, $e \downarrow m$ and for all f in $\mathbb{C}$ there exist e in $\mathcal{E}$ and m in $\mathcal{M}$ such that $f = me$.

We internalise this notion, that is, introduce internal factorisation systems for internal categories. Firstly, for an internal category, C (where $C^{\leftarrow\leftarrow}$ is defined by the pullback on the right),

$$\begin{tikzcd} C_0 \arrow[r, "e"] & C_1 \arrow[l, bend right, "d"'] \arrow[l, bend left, "c"] & C^{\leftarrow\leftarrow} \arrow[l, "m"'] \end{tikzcd} \qquad \begin{tikzcd} C^{\leftarrow\leftarrow} \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] \arrow[dr, phantom, very near start, "\lrcorner"] & C_1 \arrow[d, "c"] \\ C_1 \arrow[r, "d"] & C_0 \end{tikzcd}$$

in a finitely complete category $\mathbb{C}$, we define the *object of points* and *object of isomorphisms* as the following pullbacks (where square brackets indicate the domain of the preceding pullback projection):

$$\begin{tikzcd} \mathrm{Pt}(C) \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] \arrow[dr, phantom, very near start, "\lrcorner"] & C^{\leftarrow\leftarrow} \arrow[d, "m"] \\ C_0 \arrow[r, "e"] & C_1 \end{tikzcd} \qquad \begin{tikzcd} \mathrm{Iso}(C) \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] \arrow[dr, phantom, very near start, "\lrcorner"] & \mathrm{Pt}(C) \arrow[d, "{\pi_1[C^{\leftarrow\leftarrow}]\pi_2[\mathrm{Pt}(C)]}"] \\ \mathrm{Pt}(C) \arrow[r, "{\pi_2[C^{\leftarrow\leftarrow}]\pi_2[\mathrm{Pt}(C)]}"] & C_1 \end{tikzcd}$$

and show that the two composites of projections $\sigma = \pi_2[C^{\leftarrow\leftarrow}]\pi_2[\mathrm{Pt}(C)]\pi_1[\mathrm{Iso}(C)] : \mathrm{Iso}(C) \to C_1$ and $\sigma' = \pi_1[C^{\leftarrow\leftarrow}]\pi_2[\mathrm{Pt}(C)]\pi_1[\mathrm{Iso}(C)] : \mathrm{Iso}(C) \to C_1$ are (equivalent) subobjects of C_1, which make the following left diagram a pullback (where $C^{\leftrightarrows}$ is defined as the right pullback):

$$\begin{tikzcd} \mathrm{Iso}(C) \arrow[r, "{\langle\langle\sigma',\sigma\rangle,\langle\sigma,\sigma'\rangle\rangle}"] \arrow[d, "{(d,c)\sigma}"'] \arrow[dr, phantom, very near start, "\lrcorner"] & C^{\leftrightarrows} \arrow[d, "m\times m"] \\ C_0 \times C_0 \arrow[r, "e\times e"] & C_1 \times C_1 \end{tikzcd} \qquad \begin{tikzcd} C^{\leftrightarrows} \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] \arrow[dr, phantom, very near start, "\lrcorner"] & C^{\leftarrow\leftarrow} \arrow[d, "{(\pi_2[C^{\leftarrow\leftarrow}],\pi_1[C^{\leftarrow\leftarrow}])}"] \\ C^{\leftarrow\leftarrow} \arrow[r, "{(\pi_1[C^{\leftarrow\leftarrow}],\pi_2[C^{\leftarrow\leftarrow}])}"] & C_1 \times C_1 \end{tikzcd}$$

For two arbitrary subobjects $\alpha : A \to C_1$ and $\beta : B \to C_1$ of C_1, we call the following pullback the *object of composable morphisms* (of α and β):

$$\begin{tikzcd} B^{\leftarrow}A^{\leftarrow} \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] \arrow[dr, phantom, very near start, "\lrcorner"] & A \arrow[d, "c\alpha"] \\ B \arrow[r, "d\beta"] & C_0 \end{tikzcd}$$

We write $A^{\leftarrow\leftarrow} = A^{\leftarrow}A^{\leftarrow}$ if $\alpha = \beta$ and define an *object of composable triples*, for three subobjects of C_1, similarly.

For some subobject, $\alpha : A \to C_1$ of C_1, α is *closed under composition* (in C) if there exists a morphism $m_\alpha : A^{\leftarrow\leftarrow} \to A$ making the following diagram commute:

$$\begin{tikzcd}
A^{\leftarrow\leftarrow} \arrow[r, dashed, "m_\alpha"] \arrow[d, "\alpha\times\alpha"'] & A \arrow[d, "\alpha"] \\
C^{\leftarrow\leftarrow} \arrow[r, "m"] & C_1
\end{tikzcd}$$

Then, a pair of subobjects of C_1, $(\varepsilon : E \to C_1, \mu : M \to C_1)$ form an *internal factorisation system* on C if $\sigma \leq \varepsilon$ and $\sigma \leq \mu$ (that is, if there exist $\sigma_\varepsilon : \mathrm{Iso}(C) \to E$ and $\sigma_\mu : \mathrm{Iso}(C) \to M$ such that $\varepsilon\sigma_\varepsilon = \sigma$ and $\mu\sigma_\mu = \sigma$), ε and μ are both closed under composition, the following square is a pullback:

$$\begin{tikzcd}
M^{\leftarrow}C_1^{\leftarrow}E^{\leftarrow} \arrow[r, "m(\mu\times 1)\times 1"] \arrow[d, "1\times m(1\times\varepsilon)"'] \arrow[dr, phantom, very near start, "\lrcorner"] & C_1^{\leftarrow}E^{\leftarrow} \arrow[d, "m(1\times\varepsilon)"] \\
M^{\leftarrow}C_1^{\leftarrow} \arrow[r, "m(\mu\times 1)"] & C_1
\end{tikzcd}$$

and there exists a morphism $\tau : C_1 \to M^{\leftarrow}E^{\leftarrow}$ such that $m(\mu \times \varepsilon)\tau = 1_{C_1}$.

We internalise various properties of factorisation systems. Specifically, that the intersection of the classes is precisely the isomorphisms, that $\mathcal{E}$ and $\mathcal{M}$ respectively satisfy the right and left cancellation properties and that factorisations are unique up to isomorphism, which are respectively given by the fact that the following four squares are pullbacks:

$$\begin{tikzcd}
\mathrm{Iso}(C) \arrow[r, "\sigma_\varepsilon"] \arrow[d, "\sigma_\mu"'] \arrow[dr, phantom, very near start, "\lrcorner"] & E \arrow[d, "\varepsilon"] \\
M \arrow[r, "\mu"] & C_1
\end{tikzcd}
\qquad
\begin{tikzcd}
E^{\leftarrow\leftarrow} \arrow[r, "m_\varepsilon"] \arrow[d, "\varepsilon\times 1"'] \arrow[dr, phantom, very near start, "\lrcorner"] & E \arrow[d, "\varepsilon"] \\
C_1^{\leftarrow}E^{\leftarrow} \arrow[r, "m(1\times\varepsilon)"] & C_1
\end{tikzcd}$$

$$\begin{tikzcd}
M^{\leftarrow\leftarrow} \arrow[r, "1\times\mu"] \arrow[d, "m_\mu"'] \arrow[dr, phantom, very near start, "\lrcorner"] & M^{\leftarrow}C_1^{\leftarrow} \arrow[d, "m(\mu\times 1)"] \\
M \arrow[r, "\mu"] & C_1
\end{tikzcd}
\qquad
\begin{tikzcd}
M^{\leftarrow}\mathrm{Iso}(C)^{\leftarrow}E^{\leftarrow} \arrow[r, "m_\mu(1\times\sigma_\mu)\times 1"] \arrow[d, "1\times m_\varepsilon(\sigma_\varepsilon\times 1)"'] \arrow[dr, phantom, very near start, "\lrcorner"] & M^{\leftarrow}E^{\leftarrow} \arrow[d, "m(\mu\times\varepsilon)"] \\
M^{\leftarrow}E^{\leftarrow} \arrow[r, "m(\mu\times\varepsilon)"] & C_1
\end{tikzcd}$$

and show that these properties are satisfied by every internal factorisation system. We then induce an order on the internal factorisation systems of an internal category and show that ε and μ determine each other (up to equivalence of subobjects). We show that $(\sigma, 1_{C_1})$ forms the *trivial internal factorisation system* on an internal category, which is the top element of the order. Finally, when the base category $\mathbb{C}$ is the category **Grp**, we prove that every internal factorisation system is equivalent to the trivial one.