Presenting quotient locales

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An advantage of the pointfree approach to topology is the ability to specify frames by generators and relations. For example, the frame of real numbers $\mathcal{O}\mathbb{R}$ has a presentation with generators $(\!(p,q)\!)$ for each $p \in \mathbb{Q} \sqcup \{-\infty\}$ and $q \in \mathbb{Q} \sqcup \{\infty\}$ subject to the following relations.

- $(\!(-\infty,\infty)\!) = 1$,
- $(\!(p,q)\!) \wedge (\!(p',q')\!) = (\!(p \vee p', q \wedge q')\!)$,
- $(\!(p,q)\!) \vee (\!(p',q')\!) = (\!(p,q')\!)$ for $p \le p' < q \le q'$,
- $(\!(p,q)\!) = \bigvee_{p<p'<q'<q} (\!(p',q')\!)$.

Since sublocales correspond to quotient frames it is easy to obtain presentations of sublocales of a frame by simply adding additional relations to the original presentation. For example, adding the relations $(\!(-\infty,0)\!) = 0$ and $(\!(1,\infty)\!) = 0$ to the presentation above gives the sublocale corresponding to the closed interval $[0,1]$.

The case of quotient locales is more subtle and it is the topic of this talk. We will describe a general procedure for obtaining presentations of open or proper locale quotients from presentations of the parent locale. The result is a relatively straightforward application of the suplattice and preframe coverage theorems [1, 3], but does not appear to have been worked out explicitly before.

An open quotient of a locale X can be specified by a join-preserving closure operator on its frame of opens $\mathcal{O}X$. To present the quotient, we first ensure that the presentation for X is in the form required to apply the suplattice coverage theorem of [4, 1]. A suplattice presentation for the quotient may then be found and translated back into a frame presentation. The case of proper quotients is similar and involves interior operators that are also preframe homomorphisms.

As an example, consider the description of the circle $\mathbb{T}$ as a coequaliser in **Loc**.

$$\mathbb{R} \times \mathbb{Z} \underset{+}{\overset{\pi_1}{\rightrightarrows}} \mathbb{R} \twoheadrightarrow \mathbb{T}$$

This is the coequaliser of an open equivalence relation and thus an open quotient (see [5]). The corresponding closure operator is given by the composition of the frame map $(+)^*$ and the left adjoint $(\pi_1)_!$ and sends $(\!(p,q)\!)$ to $\bigvee_{n\in\mathbb{Z}} (\!(p-n, q-n)\!)$.

Our procedure yields a presentation for $\mathcal{O}\mathbb{T}$ with the same generators $(\!(p,q)\!)$ and the following relations.

- $(\!(-\infty,\infty)\!) = 1$,
- $(\!(p,q)\!) \wedge (\!(p',q')\!) = \bigvee_{n,n'\in\mathbb{Z}} (\!((p-n) \vee (p'-n'), (q-n) \wedge (q'-n'))\!)$.
- $(\!(p,q)\!) \vee (\!(p',q')\!) = (\!(p,q')\!)$ for $p \le p' < q \le q'$,
- $(\!(p,q)\!) = \bigvee_{p<p'<q'<q} (\!(p',q')\!)$,

This can be shown to agree with the presentation for $\mathcal{O}\mathbb{T}$ which was worked out by hand in [2].

References

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