Transfer theorems for finitely subdirectly irreducible algebras

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The aim of this work is to determine conditions under which well-studied algebraic properties transfer from the class $\mathcal{V}_{\mathrm{FSI}}$ of finitely subdirectly irreducible members of a variety $\mathcal{V}$ to the whole variety, and, in certain cases, back again.[1] The main motivation for considering $\mathcal{V}_{\mathrm{FSI}}$ rather than the class of subdirectly irreducible members of $\mathcal{V}$ is that it is often easier to establish that certain conditions hold for this larger class. Notably, if $\mathcal{V}$ has equationally definable principal meets (a common property for varieties corresponding to non-classical logics), then $\mathcal{V}_{\mathrm{FSI}}$ is a universal class [2, Theorem 1.5]. In particular, for any variety $\mathcal{V}$ of semilinear residuated lattices, $\mathcal{V}_{\mathrm{FSI}}$ is the class of totally ordered members of $\mathcal{V}$ [3].

Recall first that a class of algebras $\mathcal{K}$ has the *congruence extension property* (for short, CEP) if for any subalgebra $\mathbf{A}$ of $\mathbf{B} \in \mathcal{K}$ and congruence Θ on $\mathbf{A}$, there exists a congruence Φ on $\mathbf{B}$ satisfying $\Phi \cap A^2 = \Theta$. We prove, generalizing [4, Theorem 3.3] (see also [8, Theorem 2.3]):

Theorem. A congruence-distributive variety $\mathcal{V}$ has the congruence extension property if and only if $\mathcal{V}_{\mathrm{FSI}}$ has the CEP.

When $\mathcal{V}_{\mathrm{FSI}}$ is closed under subalgebras, this result can be reformulated in terms of commutative diagrams. A class of algebras $\mathcal{K}$ is said to have the *extension property* (for short, EP) if for any $\mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathcal{K}$, embedding $\varphi_B \colon \mathbf{A} \to \mathbf{B}$, and surjective homomorphism $\varphi_C \colon \mathbf{A} \to \mathbf{C}$, there exist a $\mathbf{D} \in \mathcal{K}$, a homomorphism $\psi_B \colon \mathbf{B} \to \mathbf{D}$, and an embedding $\psi_C \colon \mathbf{C} \to \mathbf{D}$ such that $\psi_B \varphi_B = \psi_C \varphi_C$, that is, the diagram in Figure 1(i) is commutative. A variety $\mathcal{V}$ has the CEP if and only if it has the EP, but this is not always the case for other classes, in particular, $\mathcal{V}_{\mathrm{FSI}}$. We show here that if $\mathcal{V}$ is a congruence-distributive variety such that $\mathcal{V}_{\mathrm{FSI}}$ is closed under subalgebras, then the EP and CEP for $\mathcal{V}$ and $\mathcal{V}_{\mathrm{FSI}}$ all coincide.

Recall next that a class of algebras $\mathcal{K}$ has the *amalgamation property* (for short, AP) if for any $\mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathcal{K}$ and embeddings $\varphi_B \colon \mathbf{A} \to \mathbf{B}$, $\varphi_C \colon \mathbf{A} \to \mathbf{C}$, there exist a $\mathbf{D} \in \mathcal{K}$ and embeddings $\psi_B \colon \mathbf{B} \to \mathbf{D}$, $\psi_C \colon \mathbf{C} \to \mathbf{D}$ such that $\psi_B \varphi_B = \psi_C \varphi_C$ (see Figure 1(ii)). Let us also say that $\mathcal{K}$ has the *one-sided amalgamation property* (for short, 1AP) if for any $\mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathcal{K}$ and embeddings $\varphi_B \colon \mathbf{A} \to \mathbf{B}$, $\varphi_C \colon \mathbf{A} \to \mathbf{C}$, there exist a $\mathbf{D} \in \mathcal{K}$, a homomorphism $\psi_B \colon \mathbf{B} \to \mathbf{D}$, and an embedding $\psi_C \colon \mathbf{C} \to \mathbf{D}$ such that $\psi_B \varphi_B = \psi_C \varphi_C$ (see Figure 1(iii)). It follows by [5, Lemma 2] that a variety $\mathcal{V}$ has the 1AP if and only if has the AP, but this is not always the case for other classes, in particular, $\mathcal{V}_{\mathrm{FSI}}$. We prove, generalizing [9, Theorem 9] (see also [5, Theorem 3]):

Theorem. If $\mathcal{V}$ has the congruence extension property and $\mathcal{V}_{\mathrm{FSI}}$ is closed under subalgebras, then $\mathcal{V}$ has the AP if and only if $\mathcal{V}_{\mathrm{FSI}}$ has the 1AP.

Finally, a class $\mathcal{K}$ of algebras has the *transferable injections property* (for short, TIP) if for any $\mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathcal{K}$, embedding $\varphi_B \colon \mathbf{A} \to \mathbf{B}$, and homomorphism $\varphi_C \colon \mathbf{A} \to \mathbf{C}$, there exist a $\mathbf{D} \in \mathcal{K}$, a homomorphism $\psi_B \colon \mathbf{B} \to \mathbf{D}$, and an embedding $\psi_C \colon \mathbf{C} \to \mathbf{D}$ such that $\psi_B \varphi_B = \psi_C \varphi_C$ (see Figure 1(iv)). A variety has the TIP if and only if it has the CEP and AP ([1]); more generally, we show that a class of algebras that is closed under subalgebras has the TIP if and only if it has the EP and 1AP. Our previous results then yield:

[1] An algebra $\mathbf{A}$ is *finitely subdirectly irreducible* if whenever $\mathbf{A}$ is isomorphic to a subdirect product of a non-empty finite set of algebras, it is isomorphic to one of these algebras, or, equivalently, if the least element of its congruence lattice is meet-irreducible.

$$
\begin{tikzcd}
 & \mathbf{C} \arrow[dr, hook, dashed, "\psi_C"] & \\
\mathbf{A} \arrow[ur, two heads, "\varphi_C"] \arrow[dr, hook, "\varphi_B"'] & & \mathbf{D} \\
 & \mathbf{B} \arrow[ur, dashed, "\psi_B"'] &
\end{tikzcd}
$$

(i) EP

$$
\begin{tikzcd}
 & \mathbf{C} \arrow[dr, hook, dashed, "\psi_C"] & \\
\mathbf{A} \arrow[ur, hook, "\varphi_C"] \arrow[dr, hook, "\varphi_B"'] & & \mathbf{D} \\
 & \mathbf{B} \arrow[ur, dashed, "\psi_B"'] &
\end{tikzcd}
$$

(ii) AP

$$
\begin{tikzcd}
 & \mathbf{C} \arrow[dr, hook, dashed, "\psi_C"] & \\
\mathbf{A} \arrow[ur, hook, "\varphi_C"] \arrow[dr, hook, "\varphi_B"'] & & \mathbf{D} \\
 & \mathbf{B} \arrow[ur, dashed, "\psi_B"'] &
\end{tikzcd}
$$

(iii) 1AP

$$
\begin{tikzcd}
 & \mathbf{C} \arrow[dr, hook, dashed, "\psi_C"] & \\
\mathbf{A} \arrow[ur, "\varphi_C"] \arrow[dr, hook, "\varphi_B"'] & & \mathbf{D} \\
 & \mathbf{B} \arrow[ur, dashed, "\psi_B"'] &
\end{tikzcd}
$$

(iv) TIP

Figure 1: Commutative diagrams for algebraic properties

Theorem. A congruence-distributive variety $\mathcal{V}$ such that $\mathcal{V}_{\text{FSI}}$ is closed under subalgebras has the TIP if and only if $\mathcal{V}_{\text{FSI}}$ has the TIP.

Under certain conditions, these characterizations yield effective algorithms for deciding if a finitely generated variety possesses the relevant properties. Let $\mathcal{V}$ be a congruence-distributive variety that is finitely generated by a given finite set of finite algebras such that $\mathcal{V}_{\text{FSI}}$ is closed under subalgebras. By Jónsson's Lemma ([6]), there exists and can be constructed a finite set $\mathcal{V}^*_{\text{FSI}} \subseteq \mathcal{V}_{\text{FSI}}$ of finite algebras such that each $\mathbf{A} \in \mathcal{V}_{\text{FSI}}$ is isomorphic to some $\mathbf{A}^* \in \mathcal{V}^*_{\text{FSI}}$. Hence it can be decided if $\mathcal{V}$ has the CEP by checking if each member of $\mathcal{V}^*_{\text{FSI}}$ has the CEP. Since $\mathcal{V}$ is residually small, if $\mathcal{V}$ does not have the CEP, it cannot have the AP by [7, Corollary 2.11]. Otherwise, $\mathcal{V}$ has the CEP and it can be decided if $\mathcal{V}$ has the AP (equivalently, the TIP) by checking if $\mathcal{V}_{\text{FSI}}$ has the 1AP.

References

[1] P.D. Bacsich, *Injectivity in model theory*, Colloq. Math. **25** (1972), 165–176.

[2] W.J. Blok and D. Pigozzi, *A finite basis theorem for quasivarieties*, Algebra Universalis **22** (1986), no. 1, 1–13.

[3] K. Blount and C. Tsinakis, *The structure of residuated lattices*, Internat. J. Algebra Comput. **13** (2003), no. 4, 437–461.

[4] B.A. Davey, *Weak injectivity and congruence extension in congruence-distributive equational classes*, Canad. J. Math. **29** (1977), no. 3, 449–459.

[5] G. Grätzer and H. Lakser, *The structure of pseudocomplemented distributive lattices II: Congruence extension and amalgamation*, Trans. Amer. Math. Soc. **156** (1971), 343–358.

[6] B. Jónsson, *Algebras whose congruence lattices are distributive*, Math. Scand. **21** (1967), 110–121 (1968).

[7] K.A. Kearnes, *On the relationship between AP, RS and CEP*, Proc. Amer. Math. Soc. **105** (1989), no. 4, 827–839.

[8] E.W. Kiss, *Injectivity and related concepts in modular varieties II: The congruence extension property*, Bull. Aust. Math. Soc. **32** (1985), 45–53.

[9] G. Metcalfe, F. Montagna, and C. Tsinakis, *Amalgamation and interpolation in ordered algebras*, J. Algebra **402** (2014), 21–82.