Generalized subspaces in the duality of T_D-spaces

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A space X is said to be T_D if every point $x \in X$ has an open neighborhood U such that $U - \{x\}$ is open (cf. [3]). This is a weak separation axiom, stronger than T_0 and weaker than T_1, and it plays an important role in point-free topology (see, for instance, [6]).

Actually, it can be argued that the importance of the T_D axiom is similar to that of sobriety because both concepts are, in a certain sense, dual to each other [4] — see for example the following two symmetric characterizations:

- A space X is sober if and only if there is no proper subspace inclusion $\iota\colon X \hookrightarrow Y$ such that the associated frame homomorphism $\Omega(\iota)$ is an isomorphism.
- A space X is T_D if and only if there is no proper subspace inclusion $\iota\colon Y \hookrightarrow X$ such that the associated frame homomorphism $\Omega(\iota)$ is an isomorphism.

Now, the classical adjunction

$$\begin{tikzcd} \mathsf{Top} \arrow[r, bend left, "\Omega"] \arrow[r, phantom, "\perp"] & \mathsf{Loc} \arrow[l, bend left, "\Sigma"] \end{tikzcd}$$

between topological spaces and locales restricts to an equivalence between sober spaces and spatial locales; and it was shown by Banaschewski and Pultr in [4] that there is a similar situation for the T_D-case. More precisely, there is an adjunction

$$\begin{tikzcd} \mathsf{Top_D} \arrow[r, bend left, "\Omega"] \arrow[r, phantom, "\perp"] & \mathsf{Loc}_D \arrow[l, bend left, "\Sigma'"] \end{tikzcd}$$

where Top_D denotes the category of T_D-spaces and their continuous maps, and Loc_D is a certain non-full subcategory of Loc. This adjunction restricts to an equivalence between Top_D and the subcategory of Loc_D consisting of T_D-spatial locales. Since Ω is full and faithful, one may regard Loc_D as a category of generalized T_D-spaces.

In this talk, following [1, 2], we shall discuss the basic properties of the category Loc_D, paying special attention to its regular subobject lattices (i.e., the lattices of generalized subspaces in the T_D-duality).

We will provide T_D-analogues of some well-known constructions in the theory of locales (e.g., the assembly of a frame), and explore some of their applications in point-free topology, especially in connection with T_D-spatiality. We will also stress the similarities and differences between the classical sober-spatial duality and the T_D-duality (e.g., the functorial behaviour of the assembly).

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References

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