Baker-Beynon duality beyond finite presentations

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Baker-Beynon duality is a fundamental result in the theory of abelian lattice-ordered groups (ℓ-groups, for short) and vector lattices. In [1, 2], the category of finitely-presented vector lattices is proved to be equivalent to the one of *polyhedral cones* and piecewise (homogeneous) linear maps among them —the case of ℓ-groups being slightly more complicated, as it involves polyhedral cones with *rational vertices* and maps with *integer* coefficients.

In [3] the authors propose a general framework in which many dualities, including Baker-Beynon duality, can be set. One starts with picking an arbitrary object in a category with some mild properties. The object induces a contravariant adjunction with another category to be thought of as category of *spaces*. Then a number of results, which are parametric on the arbitrary choice of the object, help characterise the fixed points of the adjunction, i.e., the ones for which the adjunction restricts to a duality. Baker-Beynon duality can be obtained from this framework setting the fixed object to be $\mathbb{R}$ and restricting to finitely generated objects. In this case the adjunction will fix only the Archimedean vector lattices (or ℓ-groups), as they are exactly the subdirect products of $\mathbb{R}$. However, if one chooses a suitable ultrapower of $\mathbb{R}$, many more objects will be left fixed by the adjunction. The reason is to be found in the following result, which is an easy consequence of quantifier elimination for vector lattices and divisible ordered groups, respectively.

Theorem 1. *For every cardinal α there exists an ultrapower of $\mathbb{R}$ on an α-regular ultrafilter, in which all linearly ordered vector lattices of cardinality smaller than α embed. The same is true for linearly ordered groups.*

We now provide more details on the general adjunctions induced by α-regular ultrapowers of $\mathbb{R}$ and the restricted dualities. Hereafter, V can be taken to be either the variety of ℓ-groups or Riesz spaces and $\mathcal{U}$ denotes invariably the α-regular ultrapower of $\mathbb{R}$ —in the appropriate language— given by Theorem 1. We denote by $\mathscr{F}^{\ell}_{\kappa}$ the free κ-generated algebra in V. Following the general framework of [3], for any $T \subseteq \mathscr{F}^{\ell}_{\kappa}$ and $S \subseteq \mathcal{U}^{\kappa}$, we define the following operators.

$$\mathbb{V}_{\mathcal{U}}(T) = \{x \in \mathcal{U}^{\kappa} \mid t(x) = 0 \text{ for all } t \in T\}, \qquad \mathbb{I}_{\mathcal{U}}(S) = \{t \in \mathscr{F}^{\ell}_{\kappa} \mid t(x) = 0 \text{ for all } x \in S\}.$$

The operators $\mathbb{V}_{\mathcal{U}}$ and $\mathbb{I}_{\mathcal{U}}$ form a Galois connection. Upon defining the appropriate notion of arrows between spaces, $\mathbb{V}_{\mathcal{U}}$ and $\mathbb{I}_{\mathcal{U}}$ can be lifted to a contravariant adjunction. A function $f\colon \mathcal{U}^n \to \mathcal{U}$ is called *definable* if there exists a term t in the language on V such that $f(p) = t(p)$ for all $p \in \mathcal{U}^n$. The definition easily generalises to functions from $S \subseteq \mathcal{U}^{\mu}$ into $S' \subseteq \mathcal{U}^{\nu}$, with μ and ν cardinals. Let $\mathsf{G}_{\mathsf{def}}$ be the category of subsets of $\mathcal{U}^{\kappa}$, with κ ranging among all cardinals, and definable maps among them.

The operators $\mathbb{V}_{\mathcal{U}}$ and $\mathbb{I}_{\mathcal{U}}$ induce functors between $\mathsf{G}_{\mathsf{def}}$ and V as follows. For any subset $S \subseteq \mathcal{U}^{\kappa}$ and for any algebra in V (assumed to be presented in the form $\mathscr{F}^{\ell}_{\kappa}/J$),

$$\mathscr{I}(S) = \mathscr{F}^{\ell}_{\kappa}/\mathbb{I}_{\mathcal{U}}(S), \quad \mathscr{V}(\mathscr{F}^{\ell}_{\kappa}/J) = \mathbb{V}_{\mathcal{U}}(J).$$

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We omit the definition of $\mathscr{I}$ and $\mathscr{V}$ on arrows, as it is more technical and not necessary in this context. By [3, Corollary 4.8] the functors $\mathscr{V}$ and $\mathscr{I}$ form a contravariant adjunction.

The fixed points of the adjunction easily correspond to the fixed points of the compositions of the operators $\mathbb{V}_{\mathcal{U}}$ and $\mathbb{I}_{\mathcal{U}}$. Regarding the algebraic side, [3, Theorem 4.15] guarantees that the ideals J for which $\mathbb{I}_{\mathcal{U}} \circ \mathbb{V}_{\mathcal{U}}(J) = J$ holds are exactly the ones that can be obtained as intersections of ideals of the form $\mathbb{I}_{\mathcal{U}}(\{a\})$ for some $a \in A$. [3, Theorem 4.15] implies that an ideal of $\mathscr{F}^{\ell}_{\kappa}$ has the form $\mathbb{I}_{\mathcal{U}}(\{a\})$ if and only if the quotient over it embeds in $\mathcal{U}$. Since both vector lattices and ℓ-groups are subdirect products of linearly ordered ones, an application of Theorem 1 proves that all the objects (of cardinality at most α) in the algebraic side of the adjunction are left fixed.

As per the fixed points in $\mathsf{G}_{\mathsf{def}}$, it is readily seen that they are the closed subspaces of $\mathcal{U}^{\kappa}$ under a Zariski-like topology given by the following closed sets:

$$\mathbb{V}_{\mathcal{U}}(T) = \{x \in \mathcal{U}^{\kappa} \mid t(x) = 0 \text{ for all } t \in T\} \text{ for } T \text{ ranging among subsets of } \mathscr{F}^{\ell}_{\kappa}. \tag{1}$$

Summing up, if V_{α} denotes the full subcategory of V whose objects have cardinality smaller than α, we obtain the following duality theorem.

Theorem 2. *There is a dual equivalence between the category V_{α} and the full subcategory K given by the closed objects in $\mathsf{G}_{\mathsf{def}}$.*

In addition to describing the dual categories to the classes of *all* Riesz spaces and ℓ-groups, Theorem 2 also enables the use of tools of non-standard analysis in the study of these structures. We give an example below, others can be found in another submitted abstract by the same authors.

Let us assume that $\mathcal{U} = \mathbb{R}^I/\mathcal{F}$ and write $[(r_i)]$ for the equivalence classes in $\mathcal{U}$. Recall that, for any $A \subseteq \mathbb{R}$ the *enlargment* of A is defined as follows:

$$[(r_i)] \in {}^*A \text{ if and only if } \{i \in I \mid r_i \in A\} \in \mathcal{F}.$$

General tools of nonstandard analysis show that basic closed set in the Zariski topology of (1), i.e. the set of the form $\mathbb{V}_{\mathcal{U}}(f)$ are enlargements of the analogous $\mathbb{V}_{\mathbb{R}}(f)$ in $\mathbb{R}^n$.

Theorem 3. *Let k be a cardinal and let J be an ideal of $\mathscr{F}^{\ell}_{\kappa}$.*

1. *$G \cong \mathscr{F}^{\ell}_{\kappa}/J$ is linearly ordered if and only if $\mathbb{V}_{\mathcal{U}}(J)$ is the closure of a point.*
2. *$G \cong \mathscr{F}^{\ell}_{\kappa}/J$ is semisimple if and only if $\mathbb{V}_{\mathcal{U}}(J)$ is the enlargement of a closed cone.*
3. *$G \cong \mathscr{F}^{\ell}_{\kappa}/J$ is finitely presented if and only if $\mathbb{V}(J)$ is the enlargement of a closed polyhedral cone – rational when G is an ℓ-group.*

References

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