(A bit more) abstract Lindenbaum lemma

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The relation between consequence relations, closure operators and closures systems is well known and so is the notion of *basis* of a closure systems (a family of closed sets which allows one to express *any* closed set as an intersection of some of its subfamily). The logical relevance of this notion is embodied in the *Lindenbaum lemma* which says that maximally consistent theories form a basis of the system of all theories (i.e., deductively closed sets) of classical propositional logic (which can be equivalently formulated as saying that every consistent theory can be extended into a maximal consistent one).

In the setting of non-classical logics the maximally consistent theories are not always sufficient to obtain the result; one has to look at, for example, the prime/complete/linear theories (depending on the logic in question). While these classes of theories are usually defined using certain logical connectives (in the mentioned examples by disjunction/negation/implication) they can usually be defined abstractly as (finitely) meet-irreducible ones.As the structurality of the underlying consequence relation is irrelevant for such a notion (i.e., it can be defined for a closure system over an arbitrary set of elements), one can formulate the following well-known crucial result of (not only) Algebraic logic:

Abstract Lindenbaum lemma Let $\mathcal{C}$ be a closure system associated to a finitary consequence relation. Then the meet-irreducible closed sets form a basis of $\mathcal{C}$.

While the finitarity restriction is crucial for its usual proof, it is not necessary: there are works (e.g. [3, 4, 5, 6, 7, 8]) proving it (or its variant for *finitely* meet-irreducible theories) for certain *infinitary structural* consequence relations (usually modal, dynamic, or fuzzy logics). The paper [1] provides a general result (covering most of the known cases) for *structural* consequence relations with a *countable* Hilbert-style axiomatization and a *strong disjunction* (see [2] for more details).

The main contribution of this paper is identifying of non-structural formulations of the necessary properties of that result and subsequent proof of its truly abstract version: We say that a consequence relation $\vdash$ on a set A with associated closure operator C and closure system $\mathcal{C}$

- is *framal,* if $\mathcal{C}$ is a frame, i.e., for each $\{X\} \cup \mathcal{Y} \subseteq \mathcal{C}$,

$$X \cap \bigvee \mathcal{Y} = \bigvee_{Y \in \mathcal{Y}} (X \wedge Y).$$

- has the *finitely generated intersection property* if for any finite sets X, Y there is a finite set U such that:

$$C(X) \cap C(Y) = C(U).$$

- is *countably axiomatizable* if there is a countable system $\mathcal{AS} \subseteq \mathcal{P}(A) \times A$ such that $X \vdash x$ iff there is tree without infinite branches labeled by elements of A such that
  - its root is labeled by x,
  - if y is a label of some of its leafs, then $y \in X$ or $\langle \emptyset, y\rangle \in \mathcal{AS}$,
  - if a non-leaf is labeled by y and Y is the set of labels of its direct predecessors, then $\langle Y, y\rangle \in \mathcal{AS}$.

Abstract Lindenbaum lemma for infinitary logics Let $\mathcal{C}$ be a closure system on a countable set A associated to a countably axiomatizable framal consequence relation with finitely generated intersection property. Then the finitely meet-irreducible closed sets form a basis of $\mathcal{C}$.

None of the three assumptions on the consequence relation can be omitted, indeed we can present examples satisfying any pair of these conditions and failing the Lindenbaum lemma (and thus also the final condition).

Let us end with a sketch of the proof. Its main tool is a binary relation $\Vdash$ on $\mathcal{P}(A)$ defined for an arbitrary consequence relation on A with associated closure operator C as:

$$X \Vdash Y \quad \text{iff} \quad \text{there is finite } Y' \subseteq Y \text{ such that } \bigcap_{y \in Y'} C(y) \subseteq C(X).$$

The two crucial facts about $\Vdash$ are ($\mathcal{C}$ is the associated closure system to C):

- If $\mathcal{C}$ is a frame, then for each sets $X, P \subseteq A$ and each *finite* set $Y \subseteq A$ we have:
$$\frac{\{X \Vdash Y \cup \{p\} \mid p \in P\} \qquad\qquad X \cup P \Vdash Y}{X \Vdash Y}.$$
- If $X \nVdash Y$ and $X \cup Y = A$, then X is a finitely meet-irreducible element of $\mathcal{C}$.

The proof is done by finding, for a given $x \notin C(X)$, a finitely meet-irreducible $X' \in \mathcal{C}$ such that $X \subseteq X'$ and $x \notin X'$. We start by enumerating all elements of an *existing countable axiomatic system* $\mathcal{AS}$ and construct a sequence of pairs $\langle X_i, Y_i \rangle$ where Y_i is finite and $X_i \nVdash Y_i$. Starting with $\langle X, \{x\} \rangle$ we in each step use the cut-like rule mentioned above (recall that we assume that $\mathcal{C}$ *is a frame*) to process the rule $\langle P_i, c_i \rangle \in \mathcal{AS}$ making sure that (roughly speaking) either we "do not have to use it" by adding c_i to X_i or that we "cannot use it" by adding some element of P_i to Y_i. Taking X' and Y' as unions of the corresponding sequences we show that $y \in C(X')$ iff $y \in X_i$ for some i which entails, using the *finitely generated intersection property,* that $X' \nVdash Y'$. Assuming that our axiomatic systems contains dummy rules $\langle \{z\}, z \rangle$ for each z we also obtain $X' \cup Y' = A$ and thus we know that X' is the set we are looking for.

References

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