Universality of the self indexing of a finitely complete category and of its monoidal generalisation

Matthew Di Meglio[1,]

School of Informatics, University of Edinburgh

For some applications of category theory in mathematics and computer science, it is useful for families of objects and morphisms of a category to be indexed not by sets but by set-like objects. Regarding the ability to reindex families as their essential characteristic, one obtains the notion of indexed categories (equivalently, fibrations). An *indexed category* $(\mathbf{S}, \mathbf{C})$ consists of

- a category $\mathbf{S}$—the *index category*—with a terminal object 1,
- for each object J of $\mathbf{S}$, a category $\mathbf{C}^J$ whose objects and morphisms are thought of as the J-indexed families of objects and morphisms of the *underlying category* $\mathbf{C}^1$,
- for each morphism $x\colon J \to K$ in $\mathbf{S}$, a functor $\Delta_x\colon \mathbf{C}^K \to \mathbf{C}^J$ which says how to reindex the K-indexed families along x,

that together form a pseudofunctor $\mathbf{S}^{\mathrm{op}} \to \mathsf{Cat}$.

An *indexing* of a category $\mathbf{C}$ is an indexed category whose underlying category is isomorphic to $\mathbf{C}$. When $\mathbf{C}$ is finitely complete, there is an indexing of $\mathbf{C}$ whose index category is also $\mathbf{C}$, whose category of J-indexed families is the slice category $\mathbf{C}/J$, and whose reindexing functors are given by chosen pullbacks; this is the *self indexing* of $\mathbf{C}$. The self indexing of $\mathbf{C}$ provides the foundation for categories internal to $\mathbf{C}$ [5, Section 15], polynomial functors in $\mathbf{C}$ [7], dependent lenses in $\mathbf{C}$ [6], and models of dependent type theory in $\mathbf{C}$. Its ubiquity suggests that it is, at least informally, the canonical indexing of $\mathbf{C}$.

Less well known is that the self indexing of a finitely complete category has a monoidal generalisation. When a symmetric monoidal category $\mathcal{V}$ has well-behaved coreflexive equalisers (i.e. has coreflexive equalisers and these are preserved by the monoidal product in each variable), there is an indexing of $\mathcal{V}$ whose index category is the category of cocommutative comonoids in $\mathcal{V}$ and whose category of J-indexed families is the category of J-comodules in $\mathcal{V}$; this is the *comonoid indexing* of $\mathcal{V}$ [for $\mathcal{V} = \mathsf{Vect}$, see 2]. If $\mathcal{V}$ is cartesian monoidal, then $\mathcal{V}$ having well-behaved coreflexive equalisers is equivalent to $\mathcal{V}$ being finitely complete; in this case, the comonoid indexing of $\mathcal{V}$ is canonically isomorphic to the self indexing of $\mathcal{V}$. The comonoid indexing of $\mathcal{V}$ gives rise to a notion of category internal to $\mathcal{V}$ that generalises the usual notion of internal category [1]; further investigation into connections with linear dependent lenses and models of linear dependent type theory is warranted. It is conceivable that there could be other indexings of nice monoidal categories that also specialise to the self indexing when the monoidal product is cartesian; to justify calling the comonoid indexing of $\mathcal{V}$ the canonical indexing of $\mathcal{V}$, we need to formalise the notion of canonicity.

Universal properties are one way to formalise notions of canonicity. The functor that sends a finitely complete category to its self indexing is right adjoint to the functor that sends a finitely-complete extensive indexed category to its underlying category; this is closely related to Moens' [4] characterisation of the $\mathbf{S}$-indexed categories that arise from finite-limit-preserving functors $F\colon \mathbf{S} \to \mathbf{C}$ [See 3, Theorem B1.4.12]. In particular, the self indexing of a finitely complete category is terminal amongst the extensive indexings of that category. Significant progress has been made towards proving a similar relationship between underlying categories and comonoid indexings in the indexed monoidal category setting.

References

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