Epimorphisms in Varieties of De Morgan Monoids*

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Let K be a variety of algebras. A homomorphism $f : \mathbf{A} \to \mathbf{B}$, with $\mathbf{A}, \mathbf{B} \in \mathsf{K}$, is called a K-*epimorphism* if, whenever $\mathbf{C} \in \mathsf{K}$ and g and h are homomorphisms from $\mathbf{B}$ to $\mathbf{C}$ such that $g \circ f = h \circ f$, then $g = h$. All surjective homomorphisms in K are K-epimorphisms. We say K has the *epimorphism surjectivity (ES) property* if the converse holds as well.

When a variety K algebraizes a logic $\vdash$, then K has the ES property if and only if $\vdash$ has the *(infinite deductive) Beth (definability) property* [5], i.e., whenever a set of variables is defined *implicitly* in terms of other variables by means of some formulas over $\vdash$, then it can also be defined *explicitly*.

A *De Morgan monoid* $\mathbf{A} = \langle A; \cdot, \wedge, \vee, \neg, e\rangle$ comprises a distributive lattice $\langle A; \wedge, \vee\rangle$, a commutative monoid $\langle A; \cdot, e\rangle$ satisfying $x \leqslant x \cdot x$, and a function $\neg \colon A \to A$, called an *involution*, such that $\mathbf{A}$ satisfies $\neg\neg x = x$ and $x \cdot y \leqslant z \iff x \cdot \neg z \leqslant \neg y$. (The derived operation $x \to y := \neg(x \cdot \neg y)$ turns $\mathbf{A}$ into a residuated lattice in sense of [3].) The class DMM of all De Morgan monoids is a variety that algebraizes the (substructural) relevance logic $\mathbf{R^t}$ of Anderson and Belnap [1].

We shall provide an overview of what is known about the ES property in varieties of De Morgan monoids. Some results are published in [8], and some will be presented in a forthcoming paper. Famously, Urquhart showed that DMM does *not* have the ES property [10]. We shall provide two sets of sufficient conditions for subvarieties that *do*. These results therefore settle natural questions about Beth-style definability for a range of substructural logics, in particular, for extensions of the relevance logic $\mathbf{R}^t$.

The first set of sufficient conditions will now be explicated. Let $\mathbf{A}$ be a De Morgan monoid. We say $\mathbf{A}$ is *negatively generated* if it is generated (as an algebra) by its set $\{a \in A \colon a \leqslant e\}$ of *negative* elements.

Recall that $F \subseteq A$ is a *deductive filter* of $\mathbf{A}$ if it is a lattice filter containing e. In this case we say F is *prime* when $F = A$ or $A \setminus F$ is a lattice ideal. Let $\mathrm{Pr}(\mathbf{A})$ be the set of prime deductive filters F of $\mathbf{A}$. We define the *depth* of F in $\mathrm{Pr}(\mathbf{A})$ to be the greatest $n \in \omega$ (if it exists) such that there is a chain in $\mathrm{Pr}(\mathbf{A})$ of the form $F = F_0 \subsetneq F_1 \subsetneq \cdots \subsetneq F_n = A$. If no such n exists, we say F has *depth* ∞ in $\mathrm{Pr}(\mathbf{A})$. We define $\mathrm{d}(\mathbf{A}) = \sup\{\mathrm{d}(F) : F \in \mathrm{Pr}(\mathbf{A})\}$. If K is a variety of De Morgan monoids, we define $\mathrm{d}(\mathsf{K}) = \sup\{\mathrm{d}(\mathbf{A}) : \mathbf{A} \in \mathsf{K}\}$. If a variety of De Morgan monoids is finitely generated, then it has finite depth (but not conversely).

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Theorem 1 ([8]). *Let* K *be a variety of De Morgan monoids of finite depth such that each finitely subdirectly irreducible member of* K *is negatively generated. Then* K *has the ES property.*

We shall see that there are $2^{\aleph_0}$ varieties of De Morgan monoids satisfying these conditions. We provide examples which show that neither of the two hypotheses in Theorem 1 can be dropped. In particular, there are $2^{\aleph_0}$ negatively generated varieties with infinite depth that do not have the ES property, despite the fact that they satisfy a 'weak' version of the ES property.

The second set of sufficient conditions concerns semilinear algebras. A De Morgan monoid is *semilinear* if it is a subdirect product of totally ordered algebras.

Theorem 2. *Every variety of negatively generated semilinear De Morgan monoids has the ES property.*

This result uses a combination of representation theorems from [4, 6, 7, 9], and we show that the class of negatively generated semilinear De Morgan monoids is a variety, in fact a locally finite one. Note that Theorem 2 generalizes the earlier finding in [2] that epimorphisms are surjective in every variety of *idempotent* ($x = x \cdot x$) De Morgan monoids, since idempotent De Morgan monoids (a.k.a. Sugihara monoids) are known to be semilinear and negatively generated.

Finally, we will give isolated examples to illustrate that Theorems 1 and 2 do not however encompass all varieties of De Morgan monoids with surjective epimorphisms. These examples will demonstrate that the property of having surjective epimorphisms is not inherited by subvarieties, while the conditions of Theorems 1 and 2 are hereditary. The main results are therefore laborsaving.

References

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