Hofmann-Mislove through the lenses of Priestley

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Let X be a sober space and $L = \mathcal{O}(X)$ the frame of open subsets of X. The Hofmann-Mislove Theorem [7] establishes that the poset of Scott-open filters of L (ordered by reverse inclusion) is isomorphic to the poset of compact saturated subsets of X (ordered by inclusion). This classic result was proved in 1981 and turned out to be an extremely useful link between domain theory and topology. Several alternative proofs of the theorem have been established since then (see, e.g., [5]). Of these, the proof by Keimel and Paseka [10] is probably the most direct and widely accepted.

There is a similar result in Priestley duality for distributive lattices [11], which establishes that the poset of filters of a bounded distributive lattice L is isomorphic to the poset of closed upsets of the Priestley space X of L. A close look at the two proofs reveals striking similarities. Indeed, it was pointed out in [2] that the latter result can be obtained from the Hofmann-Mislove Theorem. This can be seen as follows:

Let L be a bounded distributive lattice and $\mathcal{I}(L)$ the frame of ideals of L. It is well understood [9] that $\mathcal{I}(L)$ is a coherent frame, and that $\mathcal{I}$ is a functor that establishes an equivalence between the categories Dist of bounded distributive lattices and CohFrm of coherent frames. On the other hand, CohFrm is dually equivalent to the category Spec of spectral spaces [9]. Since each spectral space is sober, the Hofmann-Mislove Theorem yields that for each spectral space X, the poset of Scott-open filters of $\mathcal{O}(X)$ is isomorphic to the poset of compact saturated subsets of X. But Spec is isomorphic to the category Pries of Priestley spaces [3]. Under this isomorphism, compact saturated sets are exactly the closed upsets. Furthermore, under the equivalence between CohFrm and Dist, Scott-open filters of $\mathcal{O}(X)$ correspond to filters of the distributive lattice L of compact elements of $\mathcal{O}(X)$. Thus, the Hofmann-Mislove Theorem implies that the poset of filters of a bounded distributive lattice L is isomorphic to the poset of closed upsets of the Priestley space X of L.

We provide a new approach to the Hofmann-Mislove Theorem by showing that we can also go in the opposite direction and derive the Hofmann-Mislove Theorem by utilizing Priestley duality. Namely, let L be a frame, and let X be the Priestley space of L. Since every frame is a Heyting algebra, X is an Esakia space [4]. Moreover, since L is a complete lattice, X is extremally order-disconnected. To simplify notation, we refer to extremally order-disconnected Esakia spaces simply as *localic spaces*.

For a localic space X, let $Y = \{x \in X \mid {\downarrow}x \text{ is clopen}\}$. By [1], if X is the Priestley space of a frame L, then Y is exactly the space of points of L. Thus, L is spatial iff Y is dense in X.

The key ingredient of our proof is a characterization of Scott-open filters of L in the language of Priestley duality.

Definition 1. Let X be a localic space and C a closed upset of X. We call C a *Scott-upset* if $\min C \subseteq Y$.

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Theorem 2. *Let L be a frame, X its Priestley space, F a filter of L, and $C(F)$ its dual closed upset of X.*

(1) *F is Scott-open iff $C(F)$ is a Scott-upset.*

(2) *The poset of Scott-upsets of X is isomorphic to the poset of compact saturated subsets of Y.*

The Hofmann-Mislove Theorem is now an immediate consequence of Theorem 2. Additionally, our approach allows us to give alternate proofs for some other well-known results in domain theory and pointfree topology, including:

- Hofmann-Lawson duality between locally compact frames and locally compact sober spaces [6],
- Johnstone duality between stably locally compact frames and stably locally compact spaces [9], and
- Isbell duality between compact regular frames and compact Hausdorff spaces [8].

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