A pronilpotent look at maximal subgroups of free profinite monoids

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Profinite monoids are rich and interesting objects which are related with several topics in both algebra and logic (for instance, logic on words). In the early 2000s, Almeida established a connection between symbolic dynamics and free profinite monoids [1, 2, 3]. The work we present [8] aims to further investigate this connection in order to gain new insights into the structure of the maximal subgroups of free profinite monoids. We do this by taking a closer look at the finite nilpotent quotients of these groups, which in practical terms amounts to computing their maximal pronilpotent quotients.

Let us briefly outline the correspondence introduced by Almeida. A language $L \subseteq A^*$ is called *uniformly recurrent* when

1. it is factorial: if $w \in L$ and u is a factor of w, then $u \in L$;
2. it is extendable: if $w \in L$, then $awb \in L$ for some $a, b \in A$;
3. within L, all words are eventually unavoidable: for all $u \in L$, there exists $n \in \mathbb{N}$ such that u is a factor of every $v \in L$ with $|v| \geq n$.

In his 2007 paper [3], Almeida proved that to each uniformly recurrent language $L \subseteq A^*$ corresponds a maximal subgroup, well-defined up to isomorphism, of the free profinite monoid $\widehat{A^*}$. This group, which is a projective profinite group [10], lies inside $\overline{L} \setminus A^*$, the "infinite part" of the topological closure of L in $\widehat{A^*}$. It is known as the *Schützenberger group* of L, and can be thought of as an invariant for L [6]. In some cases, this invariant is well understood: for instance, when L is a Sturmian language, its Schützenberger group must be a free profinite group of rank 2 [5]. But not all cases are so straightforward: the Schützenberger group of the language of the Thue–Morse word is not free, not even relative to some pseudovariety of finite groups [4].

The key for understanding maximal pronilpotent quotients in the case at hand is a special kind of profinite presentations, which characterize projective objects in the category of profinite groups [9]. Generically, these presentations are of the form $\langle A \mid \varphi(a) = a,\ a \in A \rangle$, where φ is an idempotent continuous endomorphism of the free profinite group over A. In 2013, Almeida and Costa [4] obtained explicit presentations of the above form for Schützenberger groups corresponding to languages defined by *primitive substitutions* (i.e. endomorphisms of A^* whose composition matrices are primitive in the usual sense). An idempotent continuous endomorphism determining such a presentation can be computed using a *return substitution*, an important notion from symbolic dynamics which was introduced by Durand [7].

A *pronilpotent group* (respectively, *pro-p group*) is an inverse limit, in the category of compact groups, of finite discrete nilpotent groups (respectively, p-groups). In order to leverage the aforementioned profinite presentations and obtain a description of the maximal pronilpotent quotients, we rely on a number of fundamental results. First and foremost is Tate's famous characterization of projective pro-p groups, which states that they are all free (a precise statement is found e.g. in [11]). Second is the fact that the maximal pronilpotent quotient functor (left adjoint to the inclusion functor from pronilpotent groups to profinite groups) is naturally isomorphic to the product of the maximal pro-p quotient functors, where p ranges over all primes. (Essentially, for the same reason that finite nilpotent groups are isomorphic to the direct product of their Sylow subgroups.) As a result, the maximal pronilpotent quotient of a projective profinite group must be isomorphic to a product of free pro-p groups. In the specific case of the Schützenberger group corresponding to a primitive aperiodic substitution, we show that there is a transparent relationship between the rank of these pro-p factors on the one hand, and the characteristic polynomial of the composition matrix of any return substitution on the other hand. This means that all the information about the pronilpotent quotients of these groups can be neatly packaged into one single polynomial, which moreover can be effectively computed. Using the close relationship between the characteristic polynomial of a primitive substitution and those of its return substitutions (slightly strengthening a result of Durand [7]), we conclude that the characteristic polynomial of the substitution itself still carries some information about the pronilpotent quotients of the Schützenberger group.

In many cases, some features of a profinite group, such as failure of freeness, are witnessed by its pronilpotent quotients. Using this to our advantage, we devise a number of tests (i.e. necessary conditions) for freeness, both relative and absolute, of the Schützenberger groups corresponding to primitive aperiodic substitutions. These tests require little more than a quick look at the characteristic polynomial, either of the substitution itself or of one of its return substitutions. One such test, particularly easy to perform, can be succinctly phrased as follows: for the maximal subgroup corresponding to a primitive aperiodic substitution to be absolutely free, it is necessary that the product of the non-zero eigenvalues of its composition matrix (in other words, its *pseudodeterminant*) be 1 in absolute value. A notable family of substitutions failing this condition consists of primitive aperiodic substitutions of constant length, which includes the Thue–Morse substitution. In particular, it is now clear that such substitutions never produce free maximal subgroups.

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