On relative principal congruences in term quasivarieties

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In this work we introduce and study term quasivarieties. Roughly speaking, they are quasivarieties in which there are some binary terms characterizing relative principal congruences. As application we study relative compatible functions in this kind of quasivarieties.

Let A be an algebra. As usual, $\mathrm{Con}(A)$ denotes the partially ordered set of all congruences on A with respect to the inclusion. We write $\theta(a,b)$ for the smallest congruence which contains the pair (a,b): these congruences are called principal congruences. Given a quasivariety K and $A \in \mathrm{K}$ it is natural to study only those congruences of A whose quotient A/θ belongs to K. If $\theta \in \mathrm{Con}(A)$, we say that θ is a K-congruence if $A/\theta \in \mathrm{K}$. Let K be a quasivariety and $A \in \mathrm{K}$. We write $\mathrm{Con}_{\mathrm{K}}(A)$ for the partially ordered set of all K-congruences on A with respect to the inclusion. We write $\theta_{\mathrm{K}}(a,b)$ for the smallest K-congruence containing the pair (a,b): these congruences are called principal K-congruences (or relative principal congruences for short). Note that if K is a variety and $A \in \mathrm{K}$, then $\mathrm{Con}(A) = \mathrm{Con}_{\mathrm{K}}(A)$, so $\theta_{\mathrm{K}}(a,b) = \theta(a,b)$.

Given an algebra A, a function $f : A^n \to A$ is said to be compatible if any congruence of A is a congruence of the algebra (A,f). If K is a quasivariety and $A \in \mathrm{K}$, we say that f is a K-compatible operation of A if any K-congruence of A is a K-congruence of (A,f), or, equivalently, for every $\theta \in \mathrm{Con}_{\mathrm{K}}(A)$ and $a_1, b_1, \ldots, a_n, b_n \in A$ the following condition is satisfied: if $(a_i, b_i) \in \theta$ for $i = 1, \ldots n$, then $(f(a_1,\ldots,a_n), f(b_1,\ldots,b_n)) \in \theta$ (note that if K is a variety, f is K-compatible if and only if f is compatible). The logical motivation for the study of K-compatible operations comes from the notion of implicit connectives in algebraizable logics: in [3] Caicedo established a link between the implicit connectives of an algebraizable logic and the relatively compatible functions of its corresponding quasivariety obtained via the process of algebraization of Blok-Pigozzi [1].

Let A be an algebra, $f : A^n \to A$ a function and $\hat{a} = (a_1, \ldots, a_n) \in A^n$. For $i = 1, \ldots, n$ we define unary functions $f_i^{\hat{a}} : A \to A$ by $f_i^{\hat{a}}(b) := f(a_1, \ldots, a_{i-1}, b, a_{i+1}, \ldots, a_n)$. Let K be a quasivariety. There is a link between the principal K-congruences and the K-compatibility of f. More precisely, f is K-compatible if and only if for every $\hat{a} \in A^n$, $x, y \in A$ and $i = 1, \ldots, n$, $(f_i^{\hat{a}}(x), f_i^{\hat{a}}(y)) \in \theta_{\mathrm{K}}(x,y)$. Hence, a good description of the principal K-congruences may be a useful tool for the study of K-principal operations and its possible applications. If there is no ambiguity, we write relatively compatible operation instead of K-compatible operation.

In this work we are interested in quasivarieties with some particular properties. Let K be a quasivariety. We say that K is a term quasivariety if there exist an operation of arity zero e and a family of binary terms $\{t_i\}_{i \in I}$ such that for every $A \in \mathrm{K}$, $\theta \in \mathrm{Con}_{\mathrm{K}}(A)$ and $a, b \in A$ the following condition is satisfied: $(a,b) \in \theta$ if and only if $(t_i(a,b), e) \in \theta$ for every $i \in I$. In such case we say that $(e, \{t_i\}_{i \in I})$ is a pair associated to K. If a term quasivariety K is a variety, then we also say that K is a term variety.

The definition of term quasivariety is motivated by the fact that there are many quasivarieties for which the procedure to obtain a description of the relative principal congruences is exactly the same. For instance, the variety of Heyting algebras is a term variety. Indeed, if $(A, \wedge, \vee, \to, 0, 1)$ is a Heyting algebra, $\theta \in \mathrm{Con}(A)$ and $a, b \in A$, then $(a,b) \in \theta$ if and only if $((a \to b) \wedge (b \to a), 1) \in \theta$.

Let K be a a term quasivariety and $(e, \{t_i\}_{i\in I})$ a pair associated to K. For every $A \in \mathrm{K}$ we define $\Sigma = \{e/\theta : \theta \in \mathrm{Con}_{\mathrm{K}}(A)\}$. Note that Σ is a poset with the order giving by the inclusion. Moreover, Σ is a complete lattice. Let $X \subseteq A$. We define $\langle X\rangle = \bigcap_{X\subseteq e/\theta}(e/\theta)$, which is the smallest element of Σ containing X.

The main goal of this work is to describe the relative principal congruences in term quasivarieties (we also show that there exist quasivarieties which are not term quasivarieties). More precisely, we show that if K is a term quasivariety and $(e, \{t_i\}_{i\in I})$ is a pair associated to K, then for every $A \in \mathrm{K}$ and $a, b, x, y \in A$ the following condition is satisfied:

$$(x, y) \in \theta_{\mathrm{K}}(a, b) \text{ if and only if } t_j(x, y) \in \langle\{t_i(a, b)\}_{i\in I}\rangle \text{ for every } j \in I.$$

We use this description in order to characterize K-compatible functions and we give two applications of this property: 1) we give necessary conditions on K for which for every $A \in \mathrm{K}$ the K-compatible functions on A coincides with a polynomial over finite subsets of A; 2) we give a method to build up K-compatible functions. Finally, we apply the above mentioned results in order to obtain known properties about relative principal congruences and relatively compatible operations in many quasivarieties of interest for algebraic logic, as for example hemi-implicative semilattices [4, 12], RWH-algebras [5], subresiduated lattices [5, 7], semi-Heyting algebras [11], implicative semilattices [10], commutative residuated lattices [8, 9], BCK-algebras [2] and Hilbert algebras [6].

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