Algebraizability as an algebraic structure

Peter Arndt[1], Hugo Luiz Mariano[2], and Darllan Pinto[3]

[1] University of Düsseldorf, peter.arndt@uni-duesseldorf.de
[2] University of São Paulo, hugomar@ime.usp.br
[3] University of Bahia, darllan_math@hotmail.com

By *logic* we mean a pair $(\Sigma, \vdash)$ where Σ is a signature, i.e. a collection of connectives with finite arities, and $\vdash$ is an idempotent, increasing, monotone, finitary and structural consequence relation on the set $Fm_\Sigma(X)$ of formulas over Σ with a set X of variables. A *translation* between logics $(\Sigma, \vdash) \to (\Sigma', \Vdash)$ is a map $f\colon Fm_\Sigma(X) \to Fm_{\Sigma'}(X)$ induced by an arity preserving map $\Sigma \to Fm_{\Sigma'}(X)$. It is called *conservative* if $\gamma \vdash \varphi \;\Leftrightarrow\; f(\Gamma) \vdash' f(\varphi)$.

Remote Algebraizability

A *remote algebraization* of a logic L is a jointly conservative family of translations $f_i\colon L = (\Sigma, \vdash) \to (\Sigma_i, \vdash_i) = L_i$ to algebraizable logics L_i. Remote algebraization has been introduced by Bueno et al. in [**BCC**] and successfully applied to non-algebraizable, and generally badly behaved, paraconsistent logics.

Recall that a logic is algebraizable if it has a set Δ of equivalence formulas and a set $\langle \delta, \epsilon \rangle$ of pairs of formulas satisfying certain syntactic conditions given in [**BP**, Thm. 4.7].

Definition 1. *A logic is called (n,m)-algebraizable, if it admits an algebraizing pair $(\Delta, \langle \delta, \epsilon \rangle)$ for which Δ consists of at most n formulas and $\langle \delta, \epsilon \rangle$ consists of at most m pairs of formulas.*

The following construction forces a logic to become (n,m)-algebraizable:

Definition 2. *Given a logic $L = (\Sigma, \vdash)$, one defines the logic $L \otimes A_{n,m} = (\Sigma', \vdash')$ as follows:*

Σ' is obtained by adjoining binary connectives $\Delta_1, \ldots, \Delta_n$ and unary connectives $\delta_1, \ldots, \delta_m$, $\epsilon_1, \ldots, \epsilon_m$ to the signature Σ. We abbreviate $\Delta = \{\Delta_1, \ldots, \Delta_n\}$ and $\langle \delta, \epsilon \rangle = \{\langle \delta_1, \epsilon_1 \rangle, \ldots, \langle \delta_m, \epsilon_m \rangle\}$.

$\vdash'$ is the consequence relation generated by the rules of $\vdash$ and the rules making $(\Delta, \langle \delta, \epsilon \rangle)$ into an algebraizing pair.

Clearly we have an inclusion $L \to L \otimes A_{n,m}$ which is a translation, and this is a generic candidate for a remote algebraization.

Proposition 3. *A logic L admits a remote algebraization by a finite family of translations if and only if the translation $L \to L \otimes A_{n,m}$ is conservative for some $n, m \in \mathbb{N}$.*

Using this equivalence, we can characterize the finitely remotely algebraizable logics:

Theorem 4. *A logic $L = (\Sigma, \vdash)$ is remotely algebraizable by a finite family of translations if and only if one of the following conditions holds:*

(1) L has theorems.

(2) L admits no derivation of the form $\{x\} \vdash \varphi$ in which the variable x does not occur in φ.

The theorem and its proof also elucidate what are the possible obstructions to the algebraizability of a logic: On the one hand it can be missing connectives for forming an algebraizing pair – this is what we try to remedy with the construction of Def. 2. On the other hand it can be a kind of explosive behaviour, excluded by condition (2), which even prevents adding such connectives in a conservative manner!

Algebraizability as algebraic structure

We consider the category **HoLog** whose objects are logics and whose morphisms are equivalence classes of translations, where translations f, g are equivalent iff $f(\varphi) \dashv\vdash g(\varphi)$ for all φ in the domain.

The construction $L \mapsto L \otimes A_{n,m}$ of Def. 2 is part of a functor $\mathfrak{A}_{n,m} \colon \mathbf{HoLog} \to \mathbf{HoLog}$. We have natural transformations id $\to \mathfrak{A}_{n,m}$ given by the inclusions of Prop. 3 and $\mathfrak{A}_{n,m} \circ \mathfrak{A}_{n,m} \to \mathfrak{A}_{n,m}$ given by identifying the two copies of formulas of the algebraizing pair.

Theorem 5. *(1) The functor $\mathfrak{A}_{n,m}$ with these two natural transformations is a finitary monad on* **HoLog***. (2) A logic is (n,m)-algebraizable if and only if it admits an algebra structure for the monad $\mathfrak{A}_{n,m}$ (3) A logic admits at most one $\mathfrak{A}_{n,m}$-algebra structure. (4) The category of $\mathfrak{A}_{n,m}$-algebras is equivalent to the category of (n,m)-algebraizable logics and morphisms that preserve algebraizing pairs.*

From previous results of the authors one can derive that **HoLog** is locally finitely presentable. Results on monads and accessible categories then yield the following consequences:

Theorem 6. *(1) The category of (n,m)-algebraizable logics, and equivalence classes of algebraizing pair preserving translations is locally finitely presentable.*

(2) The category **HoAlg** *of algebraizable logics, and equivalence classes of algebraizing pair preserving translations is accessible.*

In particular the categories of (n,m)-algebraizable logics are equivalent to categories of models of finite limit theories, and the category **HoAlg** is equivalent to a category of models of an infinitary first order theory. This is a priori not at all clear, given the several places in which the definitions of logics and algebraizable logics refer to subsets.

Other Leibniz classes

The setup of a filtered collection of logics like the (n,m)-algebraizable logics above is precisely mirrored in Jansana's and Moraschini's definition of Leibniz class [JaMo]. In the final part of the talk we discuss how much of the above results extend to general Leibniz classes. For example for protoalgebraic logics, everything up to Thm. 5(1) and (2) goes through, but since the implication formulas witnessing protoalgebraicity are not unique, as an analog of Thm. 5(4) we obtain we obtain an equivalence with the category of protoalgebraic logics *with a chosen set of witnessing formulas.*

The analog of the construction of Def. 2 is actually a *coproduct* with a generic protoalgebraic logic, and this allows for a descent theory by which one can detect whether a logic is protoalgebraic to begin with. This is in contrast with algebraizable logics, where the construction is not a coproduct and where no such detection mechanism exists.

We finish by sketching an emerging general theory of monads and descent for Leibniz classes.

References

[**BCC**] Bueno, J., Carnielli W., Coniglio M., Finite algebraizability via possible-translations semantics, Proceedings of CombLog 04 - Workshop on Combination of Logics: Theory and Applications, 2004, p. 79-86

[**JaMo**] Jansana, R., Moraschini, T. (2021). The poset of all logics II: Leibniz classes and hierarchy. The Journal of Symbolic Logic, 1-36. Cambridge University Press (2021), http://doi.org/10.1017/jsl.2021.49

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