# Twist-structures isomorphic to modal Nelson Lattices


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In the present study, we consider extensions of constructive logic with strong Negation by means of unary modal operations. The constructive logic with strong negation has been defined by Nelson [6] and independently by Markov [5] and can be considered as a substructural logic. Nelson lattices (N3-lattices) are an algebraic semantics for this logic. They were introduced by H. Rasiowa [7] and it is known that they form a variety. An interesting result is that every Nelson lattice can be represented as a twist-structure over a Heyting algebra. A Twist structure over a lattice is a construction used by Kalman in [4] that allows us to represent an algebra as a subalgebra of a special binary power of the lattice which is obtained by considering its direct product and its order-dual. From a result of Sendlewski we know that for every Nelson lattice $\mathbf{A}$, there exists a Heyting algebra $\mathbf{H}$ such that $\mathbf{A}$ is isomorphic to a subalgebra of a twist structure over $\mathbf{H}$. Indeed, (Sendlewski + Theorem 3.1 in [1]) given a Heyting algebra $\mathbf{H} = (H, \wedge, \vee, \rightarrow, \top, \bot)$ and a Boolean filter $F$ of $\mathbf{H}$, let

$$R(\mathbf{H}, F) := \{(x, y) \in H \times H : x \wedge y = \bot \text{ and } x \vee y \in F\}. \tag{1}$$

Then we have:

1. $\mathbf{R}(\mathbf{H}, F) = (R(\mathbf{H}, F), \wedge, \vee, *, \Rightarrow, \bot, \top)$ is a Nelson lattice, where the operations are defined as follows:
   - $(x, y) \vee (s, t) = (x \vee s, y \wedge t)$,
   - $(x, y) \wedge (s, t) = (x \wedge s, y \vee t)$,
   - $(x, y) * (s, t) = (x \wedge s, (x \rightarrow t) \wedge (s \rightarrow y))$,
   - $(x, y) \Rightarrow (s, t) = ((x \rightarrow s) \wedge (t \rightarrow y), x \wedge t)$,
   - $\top = (\top, \bot)$, $\bot = (\bot, \top)$.
2. $\neg(x, y) = (y, x)$,

Given a Nelson lattice $\mathbf{A}$, there is a Heyting algebra $\mathbf{H_A}$, unique up to isomorphisms, and a unique Boolean filter $F_\mathbf{A}$ of $\mathbf{H}_A$ such that $\mathbf{A}$ is isomorphic to $\mathbf{R}(\mathbf{H_A}, F_\mathbf{A})$.

In our work, we introduce an extension of the previous twist-structure construction. We consider N3-lattices endowed with unary modal operators defined as follows. A modal N3-lattice is an algebra $\langle \mathbf{A}, \blacksquare, \blacklozenge \rangle$ such that the reduct $\mathbf{A}$ is an N3-lattice and, for all $a, b \in A$,

1. $\blacklozenge a = \neg \blacksquare \neg a$,
2. if $a^2 = b^2$ then $(\blacksquare a)^2 = (\blacksquare b)^2$ and $(\blacklozenge a)^2 = (\blacklozenge b)^2$.
3. If $(a \wedge b)^2 = \bot$ then $(\blacksquare a \wedge \blacklozenge b)^2 = \bot$.


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In addition, $\mathbf{A}$ is said to be regular if it satisfies: $\blacksquare(a \wedge b) = \blacksquare a \wedge \blacksquare b$.

Furthermore, we introduce the notion of modal Heyting algebra $M\mathbf{H}$ which is an algebra $\langle \mathbf{H}, \Box, \Diamond \rangle$ such that the reduct $\mathbf{H}$ is an Heyting algebra and

$$\text{If } a \wedge b = \bot \text{ then } \Box a \wedge \Diamond b = \bot$$

The first result we want to show is that if $\mathbf{H}$ is a modal Heyting algebra and $F$ is a Boolean filter such that

$$\text{if } a \wedge b = \bot \text{ and } a \vee b \in F \text{ then } \Box a \vee \Diamond b \in F,$$

then $\mathbf{R}(\mathbf{H}, F) = (R(\mathbf{H}, F), \wedge, \vee, *, \Rightarrow, \bot, \top, \blacksquare, \blacklozenge)$ is a Modal Nelson lattice, where the operators $\Box, \Diamond$ are defined as follows:

$$\blacksquare(x, y) = (\Box x, \Diamond y), \qquad \blacklozenge(x, y) = (\Diamond x, \Box y).$$

Also, we extend the representation of Nelson lattices in terms of Heyting algebras mentioned above to the modal context. If $\mathbf{N}$ is a modal N3 lattice, then $\mathbf{H}^* = (H, \vee^*, \wedge^*, \rightarrow^*, \neg^*, 0, 1, \Box^*, \Diamond^*)$ with $H = \{a^2 : a \in N\}$, operations $a \star^* b = (a \star b)^2$ for every binary operation $\star \in N$, $\neg^* a = (\neg a)^2$, and modal operators

$$\Box^* a = (\blacksquare a)^2, \qquad \Diamond^* a = (\blacklozenge a)^2,$$

is a modal Heyting algebra. In addition, $F = \{(a \vee \neg a)^2 : a \in N\}$ is a boolean filter of $\mathbf{H}^*$ satisfying that if $a \vee^* b \in F$ and $a \wedge^* b = 0$ then $\Box^* a \vee^* \Diamond^* b \in F$. $\mathbf{N}$ is isomorphic to $\mathbf{R}(\mathbf{H}^*, F)$ as defined in (1).

In this way, we give a more general connection between modal Nelson lattices and modal Heyting algebras that the one proposed by Jansana and Rivieccio in [3] because we do not require that modal operators satisfy monotony.

From this new connection, we are able to study the directly indecomposable modal Nelson lattices and to give some results about topological duality for *MN3*. Finally, we consider the case of Nelson lattices which further satisfy

$$\text{(Prelinearity)} \quad (x \rightarrow y) \vee (y \rightarrow x) = \top$$

usually called Nilpotent Minimum algebras (see [2]). In this context, we establish the connection between Modal Nilpotent Minimum algebras and Modal Gödel algebras which are modal Heyting algebras satisfying prelinearity.

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