Preserving joins at primes

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The aim of this talk is to show an instance of interaction between lattice theory, domain theory, and profinite monoids. Our main aim is to present an algebraic concept, *preserving joins at primes*, and develop a duality theory for it. Towards the end, we indicate two different applications of this concept in the foundations of computer science.

Implication operators. We call an *implication operator* on a bounded distributive lattice L a binary operation $\Rightarrow\colon L^{\mathrm{op}} \times L \to L$ such that, for any elements a, a', b, b' of L,

1. $\bot \Rightarrow a = \top = a \Rightarrow \top$,
2. the following two equalities hold in L:

$$(a \vee a') \Rightarrow b = (a \Rightarrow b) \wedge (a' \Rightarrow b),$$
$$a \Rightarrow (b \wedge b') = (a \Rightarrow b) \wedge (a \Rightarrow b').$$

Examples of implication operators in this sense occur frequently in the algebraic study of non-classical logics: the implication of a Heyting algebra is one example, as is the implication on an MV-algebra. Implication operators in general do not need to preserve disjunctions in the second coordinate; i.e., in logical terms, for formulas A, B and C, $(A \Rightarrow B) \vee (A \Rightarrow C)$ is stronger than $A \Rightarrow B \vee C$, and the two are not always equivalent.

Preserving joins at primes. We say an implication operator $\Rightarrow$ on a bounded distributive lattice L *preserves joins at primes* if (i) $a \Rightarrow \bot = \bot$ for any $a \in L \setminus \{\bot\}$, and, (ii) for any prime filter x of L, for any $a \in x$ and for any $b, c \in L$, there exists $a' \in x$ such that

$$a \Rightarrow (b \vee c) \leq (a' \Rightarrow b) \vee (a' \Rightarrow c).$$

To explain the name, although we do not strictly need this in what follows, we note that an equivalent formulation of this notion is the following, using the *canonical extension* L^δ of the bounded distributive lattice L [4]. An implication operator $\Rightarrow$ on L preserves joins at primes iff for any completely join-prime element x of L^δ, the following function preserves finite joins:

$$x \Rightarrow (-)\colon L \to L^\delta,$$
$$b \mapsto \bigvee \{a \Rightarrow b \mid x \leq a \in L\}.$$

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Duality. Note that an implication operator $\Rightarrow$ on L can be alternatively given by a lattice homomorphism $[\![-]\!]\colon F_{\Rightarrow}(L) \to L$, where $F_{\Rightarrow}(L)$ is the quotient of the free distributive lattice over $L \times L$ by the congruence generated by the equalities defining the notion of implication operator. This construction $L \mapsto F_{\Rightarrow}(L)$ can be made into a functor on distributive lattices. Moreover, by Priestley duality, a lattice homomorphism $[\![-]\!]\colon F_{\Rightarrow}(L) \to L$ corresponds to a continuous order-preserving map $r\colon X_L \to R(X_L)$, where X_L denotes the Priestley dual space of L, and R denotes the construction dual to $F_{\Rightarrow}$. Recall that a Priestley space X may alternatively be described via its topology of open upward closed sets, $X^{\uparrow}$, that we call the *spectral space* associated to X. We now give an explicit description of the object part of the functor R, viewed on spectral spaces. For a spectral space S, let us denote by $R(S)$ the *binary relation space* on S, i.e., the space of continuous functions from S to the *upper Vietoris space* $\mathcal{V}(S)$ of S, equipped with the compact-open topology.

In what follows, let L be a bounded distributive lattice with dual spectral space S.

Theorem 1. *The dual space of $F_{\Rightarrow}(L)$ is order-homeomorphic to the binary relation space on S.*

From this theorem and the functorial point of view on implication operators described above, one may deduce in particular the known result that implication operators $\Rightarrow$ on a bounded distributive lattice L are in one-to-one correspondence with ternary relations on the dual space X that satisfy a number of topological conditions, see e.g. [1]. Building on the above theorem, one may try to similarly characterize the implication operators that preserve joins at primes. Since the definition of preserving joins at primes is not first-order (it refers to prime filters of L), the functorial approach we outlined above for general implication operators does not go through directly. However, we do have the following. Let us say for a lattice congruence θ on $F_{\Rightarrow}(L)$ that $\Rightarrow$ *preserves joins at primes modulo* θ if, for any $a \in L \setminus \{\bot\}$, $(a \Rightarrow \bot)\theta\bot$, and for any prime filter x of L, $a \in x$, and $b, c \in L$, there is $a' \in x$ such that $a \Rightarrow (b \vee c) \leq_{\theta} (a' \Rightarrow b) \vee (a' \Rightarrow c)$. Finally, denote by $[S, S]$ the (not necessarily spectral) subspace of $R(S)$ consisting of the functions $f\colon X \to \mathcal{V}(S)$ such that $f(x)$ is a principal up-set for every $x \in S$.

Theorem 2. *The dual of the quotient of $F_{\Rightarrow}(L)$ by a congruence θ is a subspace of $[S, S]$ if and only if $\Rightarrow$ preserves joins at primes modulo θ.*

A slight generalization of this theorem also allows one to describe subspaces of $[S, T]$, where S and T are two different spectral spaces, in terms of quotients of a lattice $F_{\Rightarrow}(L, M)$ of implications between elements of L and M.

Applications. In the theory of bifinite domains [3, 2], one may actually prove that $[S, T]$ is always a bifinite domain, if S and T are. Using the above duality results, a proof of this result can be obtained by showing that, in this special setting, there is a smallest congruence, θ_j, on $F_{\Rightarrow}(L, M)$ such that $\Rightarrow$ preserves joins at primes modulo θ_j, and that this quotient is again bifinite. The dual space of the quotient by this smallest congruence θ_j is then $[S, T]$. Further, equations between domains, e.g. Scott's solution to $X \cong [X, X] \cong X \times X$, may now be analyzed dually by considering the corresponding lattices.

In the theory of regular languages and profinite monoids, where the notion of preserving joins at primes first appeared in [1], it was shown to characterize exactly those residuated families of implication operators on a Boolean algebra that are dual to a continuous binary operation.

Related work. Many of the results presented in this abstract have previously appeared in the literature, in particular in domain theory [3, 2] and topological algebra [1]. We believe the presentation and the connection between them, as mediated by the notion of join-preserving at primes, is novel. This abstract is based on parts of the two final chapters of a forthcoming textbook [5] that we are writing on duality theory, with applications to computer science.

References

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