On presheaf submonads of quantale enriched categories

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Following Lawvere's point of view that it is worth to regard metric spaces as categories enriched in the extended real half-line $[0,\infty]_+$ (see [1]), we regard both the formal ball monad and the monad that identifies Cauchy complete spaces as its algebras – which we call here the *Lawvere monad* – as submonads of the presheaf monad on the category **Met** of $[0,\infty]_+$-enriched categories. This leads us to the study of general presheaf submonads on V-**Cat**, the category of V-enriched categories, for a given a quantale V, that is, a complete lattice endowed with a symmetric tensor product $\otimes$, with unit $k \neq \bot$, commuting with joins, so that it has a right adjoint hom; this means that, for $u,v,w \in V$, $u \otimes v \leq w \;\Leftrightarrow\; v \leq \hom(u,w)$. As a category, V is a complete and cocomplete (thin) symmetric monoidal closed category.

The following well-known result (see, [2, Theorem 2.5]) plays a fundamental role in the definiton of the presheaf monad on V-**Cat**:

Theorem. *For V-categories (X,a) and (Y,b), and a V-relation $\varphi\colon X \,{\to\!\!\!\!\!\!\mapsto}\, Y$, the following conditions are equivalent:*

(i) $\varphi\colon (X,a) \,{-\!\!\!\circ\!\!\!\to}\, (Y,b)$ *is a V-distributor;*

(ii) $\varphi\colon (X,a)^{\mathrm{op}} \otimes (Y,b) \to (V,\hom)$ *is a V-functor.*

In particular, the V-categorical structure a of (X,a) is a V-distributor $a\colon (X,a) \,{-\!\!\!\circ\!\!\!\to}\, (X,a)$, and therefore a V-functor $a\colon (X,a)^{\mathrm{op}} \otimes (X,a) \to (V,\hom)$, which induces, via the closed monoidal structure of V-**Cat**, the *Yoneda V-functor* $y_X\colon (X,a) \to (V,\hom)^{(X,a)^{\mathrm{op}}}$. Thanks to the theorem above, $V^{X^{\mathrm{op}}}$ can be equivalently described as $PX := \{\varphi\colon X \,{-\!\!\!\circ\!\!\!\to}\, E \mid \varphi \text{ } V\text{-distributor}\}$. Then the structure $\widetilde{a}$ on PX is given by $\widetilde{a}(\varphi,\psi) = [\![\varphi,\psi]\!] = \bigwedge_{x\in X} \hom(\varphi(x),\psi(x))$, for every $\varphi,\psi\colon X \,{-\!\!\!\circ\!\!\!\to}\, E$, where by $\varphi(x)$ we mean $\varphi(x,*)$, or, equivalently, we consider the associated V-functor $\varphi\colon X \to V$. The Yoneda functor $y_X\colon X \to PX$ assigns to each $x \in X$ the V-distributor $x^*\colon X \,{-\!\!\!\circ\!\!\!\to}\, E$, where we identify again $x \in X$ with the V-functor $x\colon E \to X$ assigning x to the (unique) element of E. Then, for every $\varphi \in PX$ and $x \in X$, we have that $[\![y_X(x),\varphi]\!] = \varphi(x)$, as expected. In particular y_X is a fully faithful V-functor, being injective on objects (i.e. an injective map) when X is a separated V-category. We point out that $(V,\hom)$ is separated, and so is PX for every V-category X. The assignment $X \mapsto PX$ defines a functor $P\colon V\text{-}\mathbf{Cat} \to V\text{-}\mathbf{Cat}$: for each V-functor $f\colon X \to Y$, $Pf\colon PX \to PY$ assigns to each V-distributor $X \xrightarrow{\varphi} E$ the distributor $Y \xrightarrow{f^*} X \xrightarrow{\varphi} E$. It is easily checked that the Yoneda functors $(y_X\colon X \to PX)_X$

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define a natural transformation $y\colon 1 \to P$. Moreover, since, for every V-functor f, the adjunction $f_* \dashv f^*$ yields an adjunction $Pf = (\ )\cdot f^* \dashv (\ )\cdot f_* =: Qf$, Py_X has a right adjoint, which we denote by $m_X\colon PPX \to PX$. It is straightforward to check that $\mathbb{P} = (P, m, y)$ is a 2-monad on V-**Cat** – the *presheaf monad* –, which, by construction of m_X as the right adjoint to Py_X, is lax idempotent (see [3] for details).

We expand on known general characterisations of presheaf submonads and their algebras, and introduce a new ingredient – conditions of Beck-Chevalley type – which allows us to identify properties of functors and natural transformations, and, most importantly, contribute to a new facet of the behaviour of presheaf submonads. In order to do that, we will introduce the basic concepts needed to the study of V-categories and present a characterisation of the submonads of the presheaf monad using admissible classes of V-distributors which is based on [4]. Then we introduce the Beck-Chevalley conditions (BC*) which resemble those discussed in [5], with V-distributors playing the role of V-relations. In particular we show that lax idempotency of a monad $\mathbb{T}$ on V-**Cat** can be identified via a BC* condition, and that the presheaf monad satisfies fully BC*. This leads to the use of BC* to present a new characterisation of presheaf submonads.

In the remainder of the talk we will focus on the category $(V\text{-}\mathbf{Cat})^{\mathbb{T}}$ of (Eilenberg-Moore) $\mathbb{T}$-algebras, for submonads $\mathbb{T}$ of $\mathbb{P}$. We will start by reviewing some well-known results and we will conclude by presenting a new characterisation for the $\mathbb{B}$-algebras, where $\mathbb{B}$ is the *formal ball monad* on V-**Cat**, a natural generalisation of the formal ball monad on the category of (quasi-)metric spaces (cf. [6, 7]), which is constructed using the spaces of formal balls: the collections of all pairs (x, r), where $x \in X$ and $r \in [0, \infty[$, for each (quasi-)metric space X.

This talk is based on joint work with Maria Manuel Clementino. A preprint is available [8].

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