The small index property of the Fraïssé limit of finite Heyting algebras

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Consider a countable class of isomorphism types of finitely generated structures with the amalgamation property, the joint embedding property, and the hereditary property. By the *Fraïssé limit* of such a class $\mathcal{K}$, we mean the unique countable structure M whose age, i.e., the class of finitely generated substructures of M, is $\mathcal{K}$ up to isomorphism.

One pervasively studied aspect of ultrahomogenous structures—the Fraïssé limit of some classes of structures—is their automorphism groups (see, e.g., Macpherson [3]). Many studies on the automorphism groups of concrete ultrahomogeneous structures involved uniformly locally finite ones, which are necessarily ω-categorical. (For instance, the simplicity of the automorphism group of the countable atomless Boolean algebra, which is ultrahomogeneous and uniformly locally finite, was established by Anderson [1].) The present author offered in an article under review a case study on the automorphism group of a natural non-uniformly locally ultrahomogeneous structure: the Fraïssé limit L of finite Heyting algebras, whose existence follows from Maksimova's result [4] on the Craig interpolation theorem for intuitionistic logic. One main result there was that $\mathrm{Aut}(L)$ was simple.

In the present work, we show yet another important property of $\mathrm{Aut}(L)$. We equip $\mathrm{Aut}(M)$ for an arbitrary countable structure M with the so-called pointwise convergence topology, which is the topology induced as a subset of the Baire space ${}^{\omega}\omega$ if the domain of L is ω. Under this topology, every open subgroup of $\mathrm{Aut}(M)$, which is now a topological group, has countable indices. With this in mind, a topological group G is said to have the *small index property* if, conversely, every subgroup of G with a countable index is open. The topology of $\mathrm{Aut}(M)$ with the small index property is, therefore, completely determined just from its abstract group structure.

The small index property of $\mathrm{Aut}(M)$ has been shown for many ultrahomogeneous structures M. Examples relevant to the present conference include the countable atomless Boolean algebra [5] and the Fraïssé limit of finite distributive lattices [2]. By using the simplicity of $\mathrm{Aut}(L)$ and adapting Truss's argument, we obtain the following:

Theorem. The topological group $\mathrm{Aut}(L)$ has the small index property.

Following Truss, we prove this by studying the action of the automorphism group on the dual topological space of the structure. In our case, this space will be an Esakia space. Unlike the case of the countable atomless Boolean algebra, this action will not be transitive; the proof must take care of this appropriately.

Finally, we can further show the *strong* small index property, which has model-theoretic consequences on L.

References

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