Farness via Galois adjunctions and a separation theorem for uniform frames

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In classical topology, the relation of farness has been studied in [6] and used to prove an insertion result for uniform spaces in [4]. We will present the pointfree version of this notion. Instead of aproaching it in a geometrical way, we choose to describe it algebraically, in terms of Galois adjunctions. As an application, we characterize uniform frame homomorphisms and give a separation result for uniform frames.

A subset $U \subseteq L$ of a frame L is a *cover* if $\bigvee U = 1$. For each cover U of L let S_U be the *star operator*, that is

$$(x \mapsto Ux = \bigvee\{u \in U \mid u \wedge x \neq 0\})\colon L \to L$$

and let P be the *pseudocomplement operator*

$$(x \mapsto x^* = \bigvee\{y \in L \mid y \wedge x = 0\})\colon L \to L.$$

Notice that P does not depend on the cover U. While P is a self-adjoint Galois map (i.e. the pair (P,P) is a dual Galois adjunction), the star operator S_U is a left adjoint Galois map with right adjoint $\widetilde{\mathsf{S}}_U$ given by $\widetilde{\mathsf{S}}_U(y) = \bigvee\{x \in L \mid Ux \leq y\}$ (i.e. the pair $(\mathsf{S}_U, \widetilde{\mathsf{S}}_U)$ is a Galois adjunction) and we have the following diagram of adjunctions

$$L \underset{\widetilde{\mathsf{S}}_U}{\overset{\mathsf{S}_U}{\underset{\perp}{\rightleftarrows}}} L \underset{\mathsf{P}}{\overset{\mathsf{P}}{\underset{\perp^{op}}{\rightleftarrows}}} L$$

Denoting by F_U the composite $\mathsf{P}\mathsf{S}_U$ (which can be proved to be equal to $\widetilde{\mathsf{S}}_U\mathsf{P}$), elements $a, b \in L$ are *U-far* if $a \leq \mathsf{F}_U(b)$ (or, equivalently, $b \leq \mathsf{F}_U(a)$).

Since we are interested in farness in uniform frames, we recall some notions that were first studied in [5] (see [3] for more information). A *uniformity* on a frame L is a system $\mathcal{U}$ of covers such that

(U1) $U \in \mathcal{U}$ and $U \leq V$ implies $V \in \mathcal{U}$,

(U2) $U, V \in \mathcal{U}$ implies $U \wedge V \in \mathcal{U}$,

(U3) for every $U \in \mathcal{U}$ there is a $V \in \mathcal{U}$ such that $VV \leq U$, and

(U4) for every $a \in L$, $a = \bigvee\{b \mid b \lhd_{\mathcal{U}} a\}$

where $U \wedge V = \{u \wedge v \mid u \in U, v \in V\}$, $VV = \{\mathsf{S}_V(v) \mid v \in V\}$ and we write $b \lhd_{\mathcal{U}} a$ if $\mathsf{S}_U(b) \leq a$ for some $U \in \mathcal{U}$. Without (U4), we say $\mathcal{U}$ is a *preuniformity*. A frame homomorphism $f\colon L \to M$ is a *uniform homomorphism* $(L, \mathcal{U}) \to (M, \mathcal{V})$ if $h[U] \in V$ for every $U \in \mathcal{U}$.

As an example, we can consider the frame of reals $\mathfrak{L}(\mathbb{R})$ presented by generators $(p, —)$ and $(—, p)$ for all rationals p and a given set of relations ([3]). This frame carries its metric uniformity (see [2]). Thus, for a frame L with a (pre)uniformity $\mathcal{U}$, we say a real-valued function $f\colon \mathfrak{L}(\mathbb{R}) \to L$ is *uniformly continuous* if it is a uniform frame homomorphism with respect to the metric uniformity of $\mathfrak{L}(\mathbb{R})$ and $\mathcal{U}$. We have the following characterization for real-valued uniform frame homomorphsims:

Theorem. *Let $(L,\mathcal{U})$ be a (pre)uniform frame. The following are equivalent for any frame homomorphism $f\colon \mathfrak{L}(\mathbb{R}) \to L$:*

(i) *f is uniformly continuous.*

(ii) *For every $\delta \in \mathbb{Q}^+$, there is some $U \in \mathcal{U}$ such that $f(—, r)$ and $f(s, —)$ are U-far for all $r, s \in \mathbb{Q}$ such that $s - r > \frac{1}{\delta}$.*

As an application one can obtain an Urysohn-type separation result for uniform frames, namely:

Theorem. *Let $\mathcal{U}$ be a (pre)uniformity on a frame L. If a and b are U-far, for some $U \in \mathcal{U}$, then there is a bounded uniformly continuous $f : \mathfrak{L}(\mathbb{R}) \to L$ such that $f(0, —) \leq a^*$ and $f(—, 1) \leq b^*$.*

We will present the proof of this result which features a purely algebraic (order-theoretic) construction; it mainly relies on the Galois adjunction that defines the relation of farness.

This talk is based on the preprint [1] and it is a joint work with Jorge Picado.

References

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