Farness via Galois adjunctions and a separation theorem for uniform frames

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In classical topology, the relation of farness has been studied in [6] and used to prove an insertion result for uniform spaces in [4]. We will present the pointfree version of this notion. Instead of aproaching it in a geometrical way, we choose to describe it algebraically, in terms of Galois adjunctions. As an application, we characterize uniform frame homomorphisms and give a separation result for uniform frames.

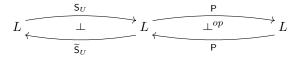
A subset $U \subseteq L$ of a frame L is a cover if $\bigvee U = 1$. For each cover U of L let S_U be the star operator, that is

$$(x \mapsto Ux = \bigvee \{ u \in U \mid u \land x \neq 0 \}) \colon L \to L$$

and let P be the *pseudocomplement operator*

$$(x \mapsto x^* = \bigvee \{ y \in L \mid y \land x = 0 \}) \colon L \to L.$$

Notice that P does not depend on the cover U. While P is a self-adjoint Galois map (i.e. the pair (P, P) is a dual Galois adjunction), the star operator S_U is a left adjoint Galois map with right adjoint \widetilde{S}_U given by $\widetilde{S}_U(y) = \bigvee \{x \in L \mid Ux \leq y\}$ (i.e. the pair (S_U, \widetilde{S}_U) is a Galois adjunction) and we have the following diagram of adjunctions



Denoting by F_U the composite PS_U (which can be proved to be equal to $\widetilde{\mathsf{S}}_U\mathsf{P}$), elements $a, b \in L$ are U-far if $a \leq \mathsf{F}_U(b)$ (or, equivalently, $b \leq \mathsf{F}_U(a)$).

Since we are interested in farness in uniform frames, we recall some notions that were first studied in [5] (see [3] for more information). A *uniformity* on a frame L is a system \mathcal{U} of covers such that

- (U1) $U \in \mathcal{U}$ and $U \leq V$ implies $V \in \mathcal{U}$,
- (U2) $U, V \in \mathcal{U}$ implies $U \wedge V \in \mathcal{U}$,
- (U3) for every $U \in \mathcal{U}$ there is a $V \in \mathcal{U}$ such that $VV \leq U$, and
- (U4) for every $a \in L$, $a = \bigvee \{b \mid b \triangleleft_{\mathcal{U}} a\}$

where $U \wedge V = \{u \wedge v \mid u \in U, v \in V\}$, $VV = \{S_V(v) \mid v \in V\}$ and we write $b \triangleleft_{\mathcal{U}} a$ if $S_U(b) \leq a$ for some $U \in \mathcal{U}$. Without (U4), we say \mathcal{U} is a *preuniformity*. A frame homomorphism $f : L \to M$ is a *uniform homomorphism* $(L, \mathcal{U}) \to (M, \mathcal{V})$ if $h[U] \in V$ for every $U \in \mathcal{U}$.

As an example, we can consider the frame of reals $\mathfrak{L}(\mathbb{R})$ presented by generators (p, -)and (-, p) for all rationals p and a given set of relations ([3]). This frame carries its metric uniformity (see [2]). Thus, for a frame L with a (pre)uniformity \mathcal{U} , we say a real-valued function $f: \mathfrak{L}(\mathbb{R}) \to L$ is uniformly continuous if it is a uniform frame homomorphism with respect to the metric uniformity of $\mathfrak{L}(\mathbb{R})$ and \mathcal{U} . We have the following characterization for real-valued uniform frame homomorphisms:

Theorem. Let (L, \mathcal{U}) be a (pre)uniform frame. The following are equivalent for any frame homomorphism $f: \mathfrak{L}(\mathbb{R}) \to L$:

- (i) f is uniformly continuous.
- (ii) For every $\delta \in \mathbb{Q}^+$, there is some $U \in \mathcal{U}$ such that f(-,r) and f(s,-) are U-far for all $r, s \in \mathbb{Q}$ such that $s-r > \frac{1}{\delta}$.

As an application one can obtain an Urysohn-type separation result for uniform frames, namely:

Theorem. Let \mathcal{U} be a (pre)uniformity on a frame L. If a and b are U-far, for some $U \in \mathcal{U}$, then there is a bounded uniformly continuous $f : \mathfrak{L}(\mathbb{R}) \to L$ such that $f(0, -) \leq a^*$ and $f(-, 1) \leq b^*$.

We will present the proof of this result which features a purely algebraic (order-theoretic) construction; it mainly relies on the Galois adjunction that defines the relation of farness.

This talk is based on the preprint [1] and it is a joint work with Jorge Picado.

References

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