Difference–restriction algebras of partial functions with operators: discrete duality and completion

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We exhibit an adjunction between a category of abstract algebras of partial functions and a category of set quotients. The algebras are those atomic algebras representable as a collection of partial functions closed under relative complement and domain restriction; the morphisms are the complete homomorphisms. This generalises the discrete adjunction between the atomic Boolean algebras and the category of sets. We define the compatible completion of a representable algebra, and show that the monad induced by our adjunction yields the compatible completion of any atomic representable algebra. As a corollary, the adjunction restricts to a duality on the compatibly complete atomic representable algebras, generalising the discrete duality between complete atomic Boolean algebras and sets. We then extend these adjunction, duality, and completion results to atomic representable algebras equipped with arbitrary additional completely additive and compatibility preserving operators.

The paper [2] that this talk corresponds to is a sequel to *Difference–restriction algebras of partial functions: axiomatisations and representations* [1].

Collections of partial functions, together with operations on those functions (think, for example, of composition) can be studied as algebraic structures whose elements are the functions and operations are the given operations. In this framework, each distinct choice σ of operations specifies a distinct class of algebras to be studied: we say an algebra (of the correct type) is **representable** if it is isomorphic to a collection of partial functions equipped with the operations in σ, and then we may study the class of representable algebras. And indeed many of these classes have been investigated, particularly in terms of their axiomatisability and in terms of complexity questions. (See [6, §3.2] for a guide to this literature.)

Recently, a number of categorical duality results for classes of partial function algebras have started to appear ([4, 5, 3, 7] and others), in the spirit of Stone duality between Boolean algebras and Stone spaces. This talk concerns a project aiming to develop a general unified theory for dualities of partial function algebras. In this we are guided by the example provided by the duality between Boolean algebras with operators and descriptive general frames, specifically the modular nature of that duality, where arbitrary operations satisfying certain conditions can be appended to a base duality.

For our base category—our analogue of Boolean algebras—we use the isomorphs of the following algebras.

Definition. An **algebra of partial functions** of the signature $\{-, \triangleright\}$ is a universal algebra $\mathfrak{A} = (A, -, \triangleright)$ where the elements of the universe A are partial functions from some (common) set X to some (common) set Y and the interpretations of the symbols are given as follows:

- The binary operation $-$ is **relative complement**:

$$f - g := \{(x, y) \in X \times Y \mid (x, y) \in f \text{ and } (x, y) \notin g\}.$$

- The binary operation $\triangleright$ is **domain restriction**. It is the restriction of the second argument to the domain of the first; that is:

$$f \triangleright g := \{(x, y) \in X \times Y \mid x \in \mathrm{dom}(f) \text{ and } (x, y) \in g\}.$$

In [1] we axiomatised the class of representable algebras for the signature $\{-, \triangleright\}$, and we also axiomatised the smaller class of *completely representable* algebras, for the same signature. An algebra is **completely representable** if it can be embedded into an algebra of partial functions in such a way that arbitrary cardinality joins (whenever they exist) are transformed into unions. (Here, the partial order on representable algebras is defined by $a \leq b \iff a \triangleright b = a$ or equivalently $a \leq b \iff b - (b - a) = a$ and corresponds to inclusion.) The completely representable $\{-, \triangleright\}$-algebras turn out to be precisely the representable algebras that are *atomic*.

In this talk we develop duality theory for the category of completely representable $\{-, \triangleright\}$-algebras (with morphisms the complete, i.e. arbitrary join-preserving, homomorphisms). Classes of completely representable algebras are good candidates for 'discrete' dualities, that is, dualities in which the opposite category is absent any topological content. This is indeed the case here. We first exhibit an *adjunction* between the category of completely representable algebras and a category of set quotients. This generalises the discrete adjunction between the atomic Boolean algebras and the category of sets. Briefly, the adjoint of an algebra is the set of its atoms together with the quotient corresponding to the 'have the same domain' equivalence ($a \triangleright b = b$ and $b \triangleright a = a$) on partial functions.

We then define the *compatible completion* of a representable algebra, and show that the monad induced by our adjunction yields the compatible completion of any atomic representable algebra. As a corollary, the adjunction restricts to a duality on the compatibly complete atomic representable algebras, generalising the discrete duality between complete atomic Boolean algebras and sets.

Finally, we extend these adjunction, duality, and completion results to completely representable algebras equipped with arbitrary additional completely additive and compatibility preserving operators. These generalise results for atomic Boolean algebras with completely additive operators.

References

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