Canonical extensions and the frame of fitted sublocales

Anna Laura Suarez[1,*] and Tomáš Jakl[2]

[1] Université Côte d'Azur, Nice, France
annalaurasuarez993@gmail.com

[2] University of Cambridge, UK
Tomas.Jakl@cl.cam.ac.uk

1 Sublocales of the frame of strongly exact filters

When we see a frame L as a pointfree space its *sublocales* are its pointfree subspaces. Sublocales are certain subsets of L, and these form a coframe $\mathsf{S}(L)$ when ordered under set inclusion. We have open sublocales, closed sublocales, and fitted sublocales; they correspond, respectively, to open, closed, and saturated subsets of a space. For a frame L we have an order anti-isomorphism between the collection of fitted sublocales $\mathsf{Fitt}(L)$ and the frame $\mathsf{SEFilt}(L)$ of the *strongly exact* filters (see [3]).

I will show in this talk that the frame of strongly exact filters has some distinguished sublocales, which correspond to special subcolocales of $\mathsf{Fitt}(L)$ because of the result above. We have a diagram of sublocale inclusions as follows.

$$\begin{tikzcd}
\mathsf{MCFilt}(L) \arrow[r, "\subseteq"] & \mathsf{EFilt}(L) \arrow[dr, "\subseteq"] & & \\
& & \mathsf{SEFilt}(L) \arrow[r, "\subseteq"] & \mathsf{Filt}(L) \\
\mathsf{MCPFilt}(L) \arrow[r, "\subseteq"] & \mathsf{MSOFilt}(L) \arrow[ur, "\subseteq"] & &
\end{tikzcd}$$

Here $\mathsf{MCFilt}(L)$ is the frame of meets of *closed* filters, that is, filters of the form $\{x \in L : x \vee a = 1\}$ for some $a \in L$. The frame $\mathsf{EFilt}(L)$ is the frame of *exact* filters, a notion that appears in [3] and is in a certain sense dual to that of strongly exact filter. The frame $\mathsf{MCPFilt}(L)$ is the frame of meets of completely prime filters, and the frame $\mathsf{MSOFilt}(L)$ is the frame of meets of Scott-open filters.

The diagram above corresponds to the following diagram of subcolocale inclusions.

$$\begin{tikzcd}
\mathit{fit}[S_{\mathfrak{c}}(L)] \arrow[r, "\subseteq"] & \mathit{fit}[\mathsf{S}_b(L)] \arrow[dr, "\subseteq"] & \\
& & \mathsf{Fitt}(L) \\
\mathit{fit}[\mathsf{sp}[\mathsf{S}(L)]] \arrow[r, "\subseteq"] & \mathit{fit}[\mathsf{S}_k(L)] \arrow[ur, "\subseteq"] &
\end{tikzcd}$$

*Speaker.

Here $\mathsf{S}_{\mathfrak{c}}(L)$ is the collection of joins of closed sublocales (see [4]); $\mathsf{S}_b(L)$ is the coframe of joins of complemented sublocales (whose structure has been recently studied in [5]). $\mathsf{S}_k(L)$ is the collection of joins of compact sublocales, and $\mathsf{sp}[\mathsf{S}(L)]$ is the coframe of spatial sublocales. Finally, $\mathit{fitt} : \mathsf{S}(L) \to \mathsf{S}(L)$ denotes the *fitting* operator (see [2]), which is the closure coming from $\mathsf{Fitt}(L) \subseteq \mathsf{S}(L)$ seen as a closure system.

2 The connection with canonical extensions

Canonical extensions for frames have been recently studied in [1]. The canonical extension of a frame is got as a *polarity*. A *polarity* is a complete lattice $\mathsf{Pol}(X, Y, R)$ such that X and Y are sets and $R \subseteq X \times Y$ is a relation, and such that we have two canonical maps $f_X : X \to \mathsf{Pol}(X, Y, R)$ and $f_Y : Y \to \mathsf{Pol}(X, Y, R)$ satisfying a certain universal property. The sublocales of $\mathsf{Fitt}(L)$ above are all instances of polarities. In general, in fact, we have that for a collection $\mathcal{S} \subseteq \mathsf{S}(L)$ of sublocales and for the collection $\mathsf{Op}(L)$ of open sublocales

$$\mathsf{Pol}(\mathcal{S}, \mathsf{Op}(L), \subseteq) = \mathit{fitt}[\mathfrak{J}(\mathcal{S})],$$

where $\mathfrak{J}$ denotes closure under all joins. The theory of polarities also helps us see a symmetry between fitted and closed sublocales. We have in general that

$$\mathsf{Pol}(\mathcal{S}, \mathsf{Cl}(L), \subseteq) = cl[\mathfrak{J}(\mathcal{S})].$$

This means that the structures above all enjoy a universal property, all variations of the defining universal property of the canonical extension of a frame. This also leads to the following result exhibiting a symmetry between the closures *fit* and *cl*, which were recently compared in [2]. Here for a frame L the expression $\mathfrak{B}(L)$ denotes the Booleanization of L. For a frame L we have:

1. $\mathsf{Pol}(\mathsf{Cl}(L), \mathsf{Op}(L), \subseteq) \cong \mathit{fit}[\mathsf{S}_c(L)] = \mathfrak{B}(\mathsf{Fitt}(L))$;
2. $\mathsf{Pol}(\mathsf{Op}(L), \mathsf{Cl}(L), \subseteq) \cong cl[\mathsf{Op}(L)] = \mathfrak{B}(\mathsf{Cl}(L)) \cong \mathfrak{B}(L)$.

Since this is work in progress, in the end of the talk I will outline some open questions, especially related to how much more interaction we can find between the theory of polarities and the closure systems within $\mathsf{S}(L)$.

References

[1] Jakl, T. Canonical extensions of locally compact frames. *Topology And Its Applications.* **273** pp. 106976 (2020)

[2] Clementino, M., Picado, J. & Pultr, A. The Other Closure and Complete Sublocales. *Applied Categorical Structures.* **26** (2018,2)

[3] Moshier, M., Pultr, A. & Suarez, A. Exact and Strongly Exact Filters. *Applied Categorical Structures.* **28**, 907-920 (2020)

[4] Picado, J., Pultr, A. & Tozzi, A. Joins of closed sublocales. *Houston Journal Of Mathematics.* **45** pp. 21-38 (2019)

[5] Arrieta, I. On joins of complemented sublocales. *Algebra Universalis.* **83** (2022)