Locally non-separating sublocales and Peano compactifications

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In [1], Curtis introduced the concept of a *locally non-separating* remainder in order to study the hyperspace of a non-compact space X. Using the property of a locally non-separating remainder, Curtis established the conditions under which a *Peano* compactification of a connected space X would exist. In this talk, we discuss the analog of the concept of locally non-separating sets, in frames. We begin with a discussion of properties of sublocales, after which we define a locally non-separating sublocale and conclude by providing a generalisation for a special case of Curtis's result.

1 Some notes on sublocales

We recall the definition of the *supplement* and *difference* of a sublocale amongst other properties from Plewe [2]. The following results will be discussed:

Lemma 1.1. *If T is a complemented sublocale of a frame L, and S is any sublocale of L, then $S \setminus (L \setminus T) = S \cap T$.*

Lemma 1.2. *Let A be a sublocale of L, and for any subset B of L, let $\{\mathfrak{o}(b) \mid b \in B\}$ be a collection of open sublocales in L. Then*

$$(\bigvee_{b \in B} \mathfrak{o}(b)) \setminus A = \bigvee_{b \in B} (\mathfrak{o}(b) \setminus A).$$

Lemma 1.3. *S is a dense sublocale of L if and only if S meets every non-trivial open sublocale of L.*

The results which follow, are concerned with useful properties of the images of sublocales under the right adjoint of a given frame homomorphism. For the purpose of this talk, $h_* : M \to L$ shall denote the right adjoint of $h : L \to M$, where h is a frame homomorphism.

Proposition 1.4. *If $h : L \to M$ is any frame homomorphism, $a \in L$, and T is a sublocale of M, then :*

(1) $h_*(T) \subseteq \uparrow a \iff T \subseteq \uparrow h(a)$,

(2) $h_*(T) \cap \uparrow a = \{1_L\} \iff T \cap \uparrow h(a) = \{1_M\}$,

(3) $h_*(T) \subseteq \mathfrak{o}(a) \iff T \subseteq \mathfrak{o}(h(a))$.

2 Locally non-separating sublocales

We shall assume that L is a locally connected frame

Definition 2.1. A non-trivial sublocale A of L is called *locally non-separating sublocale* in L, if whenever $\{1\} \neq U \subseteq L$ is an open connected sublocale then $U \setminus A \neq \{1\}$ and $U \setminus A$ is connected as a sublocale.

The following proposition is required to show that every non-trivial sublocale of a locally non-separating sublocale is locally non-separating.

Proposition 2.2. *Let S and T be sublocales of L. $S \subseteq \overline{T}$ if and only if for every non-trivial open sublocale $\mathfrak{o}(a)$ of L such that $\mathfrak{o}(a) \cap S \neq \{1\}$ then $\mathfrak{o}(a) \cap T \neq \{1\}$.*

Proposition 2.3. *Let A and B be sublocales of L such that $\{1\} \neq B \subseteq A$. If A is locally non-separating in L then B is locally non-separating in L.*

Theorem 2.4. *Let $B \subseteq L$ be a base of L consisting of connected elements. Suppose $A \neq \{1\}$ is a sublocale of L and that $\mathfrak{o}(b) \setminus A \neq \{1\}$ is connected for each $b \in B$. Then A is locally non-separating in L.*

3 A Peano compactification with a locally non-separating remainder

Curtis established, in [1], that a connected space X having a Peano compactification with a specified *locally non-separating remainder* is equivalent to the space X being S-metrizable. We provide a generalisation of the above result under the assumption of L being a *regular continuous* frame. In order to do so, we first define a *locally non-separating remainder* of a frame.

Definition 3.1. Let S be a sublocale of L. Then $L \setminus S$ is called a *locally non-separating remainder* if $L \setminus S$ is locally non-separating in L.

Recall that a frame homomorphism $h : L \to M$ is said to be *open* precisely when $h_*(U)$ is an open sublocale of L, for every open sublocale U of M.

Proposition 3.2. *If $h : L \to M$ is an onto frame homomorphism and $h_*(M)$ is an open sublocale of L, then h is an open map.*

Proposition 3.3. *Let $h : L \to M$ be a compactification of M, where M is non-compact and regular continuous, then $h_*(M) = \mathfrak{o}(a)$, if $a = \bigvee\{h_*(x) \mid x << 1_M\}$, and hence h is an open map.*

The following theorem is the main result:

Theorem 3.4. *Suppose that M is a non-compact, connected and regular continuous frame. Then M has a Peano compactification $h : L \to M$ with a locally non-separating remainder $L \setminus h_*(M)$ if and only if M is S-metrizable.*

References

[1] D. W. Curtis, *Hyperspaces of Noncompact Metric Spaces*, Compositio Math. 40 (1980), 139 - 152.

[2] T. Plewe, *Quotient Maps of Locales*, Appl. Categ. Structures 8 (2000) 17–44.