Regular categories and soft sheaf representations

Marco Abbadini[1] and Luca Reggio[2,*]

[1] Dipartimento di Matematica, Università degli Studi di Salerno, Italy
mabbadini@unisa.it

[2] Department of Computer Science, University College London, United Kingdom
l.reggio@ucl.ac.uk

The aim of this work is to study (soft) sheaf representations of objects of regular categories. Sheaf representations of universal algebras have been investigated since the 1970s, see e.g. [3, 5, 9, 13], inspired by several results for rings and modules obtained in the 1960s [4, 8, 12].

In particular it was observed that, for a universal algebra A, any distributive lattice of pairwise commuting congruences on A induces a sheaf representation of A [13] (i.e. a sheaf whose algebra of global sections is isomorphic to A). The sheaf representations over *stably compact* spaces [10] arising in this way were characterised by Gehrke and van Gool [6], who recognised the central role of the notion of softness [7]. A sheaf over a space X is *soft* if, for all compact saturated[1] subsets $K \subseteq X$, every (continuous) section over K can be extended to a global section. In [6], a bijection was established between (isomorphism classes of) soft sheaf representations of an algebra A over a stably compact space X, and frame homomorphisms from the (co-compact) dual frame of X to a frame of commuting congruences on A.

We generalise the previous result by replacing varieties of algebras—in which sheaves take values—with any *regular* category [1], i.e. a category C such that:

(i) C has finite limits.

(ii) C has (regular epi, mono) factorisations, i.e. every arrow f in C can be written as $f = m \circ e$ where e is a regular epimorphism and m a monomorphism.

(iii) Regular epimorphisms in C are stable under pullbacks along any morphism.

Regular categories are a non-additive generalisation of Abelian categories. Examples of regular categories include most "algebraic-like" categories such as varieties and quasi-varieties of (possibly infinitary) algebras, any topos, the categories of Stone spaces and of compact Hausdorff spaces (and their opposite categories), and the opposite of the category of topological spaces.

If A is an object of a regular category C, the role of the "congruence lattice" of A is played by the category $\mathrm{RegEpi}\,A$ of regular epimorphisms with domain A. Commuting congruences then correspond to *ker-commuting* objects of $\mathrm{RegEpi}\,A$ [2]. Fix an arbitrary complete lattice P. The functor category $[P^{\mathrm{op}}, \mathrm{RegEpi}\,A]$ can be identified with the large preorder of monotone maps $P^{\mathrm{op}} \to \mathrm{RegEpi}\,A$, with respect to the pointwise preorder. The codomain functor $\gamma\colon \mathrm{RegEpi}\,A \to \mathsf{C}$ induces a "direct image" functor

$$\gamma_*\colon [P^{\mathrm{op}}, \mathrm{RegEpi}\,A] \to [P^{\mathrm{op}}, \mathsf{C}], \quad H \mapsto \gamma \circ H.$$

Preservation of certain infima or suprema under a monotone map $H\colon P^{\mathrm{op}} \to \mathrm{RegEpi}\,A$ then corresponds to "sheaf-like" properties of the functor $\gamma_* H$. To make this precise we introduce

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*Speaker.

[1] A subset of a topological space is *saturated* if it is an intersection of open sets. Whenever X is locally compact and Hausdorff, "compact saturated" can be replaced with "closed" in the definition of softness.

the following notion of $\mathcal{K}$-sheaf, inspired by the work of Lurie [11, Chapter 7]. Intuitively, a $\mathcal{K}$-sheaf is a sheaf defined on the compact (or, more generally, compact saturated) subsets of a space, rather than on the open ones.

Definition. A C-*valued* $\mathcal{K}$-*sheaf* on P is a functor $F\colon P^{\mathrm{op}} \to \mathsf{C}$ such that:

(K1) $F(\bot_P)$ is a subterminal object of C.

(K2) $\forall p, q \in P$, the image under F of the diagram $\begin{array}{ccc} p \wedge q & \longrightarrow & p \\ \downarrow & & \downarrow \\ q & \longrightarrow & p \vee q \end{array}$ in P is a pullback in C.

(K3) F preserves directed colimits.

Theorem. *Let $H\colon P^{\mathrm{op}} \to \mathrm{RegEpi}\, A$ be a monotone map whose image consists of pairwise ker-commuting elements. The following statements are equivalent:*

1. *H preserves finite infima and non-empty suprema.*
2. *$\gamma_* H\colon P^{\mathrm{op}} \to \mathsf{C}$ is a $\mathcal{K}$-sheaf.*

Let M be the (large) sub-preorder of $[P^{\mathrm{op}}, \mathrm{RegEpi}\, A]$ consisting of those maps that preserve finite infima and arbitrary suprema, and whose images consist of pairwise ker-commuting elements. The previous theorem induces an isomorphism of categories between M and a category of soft $\mathcal{K}$-sheaf representations of A over P (a $\mathcal{K}$-sheaf is *soft* if all arrows in its image are regular epimorphisms). If C is *Barr-exact* (e.g. if C is a variety of algebras), regular epimorphisms with domain A can be replaced with (internal) equivalence relations on A, and ker-commuting elements of $\mathrm{RegEpi}\, A$ with commuting equivalence relations.

Suppose that C is a variety of algebras (more generally, a complete and cocomplete regular category in which finite limits commute with filtered colimits). If the lattice P is nice enough, the category of (soft) $\mathcal{K}$-sheaves $P^{\mathrm{op}} \to \mathsf{C}$ is equivalent to a category of ordinary (soft) sheaves. E.g., if P is the lattice of compact saturated subsets of a stably compact space X, then $\mathcal{K}$-sheaves over P correspond to ordinary sheaves on X. We thus recover the main result of [6].

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