Cohomological refinements of k-consistency and k-equivalence

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The seminal work of Kochen and Specker [11] showed that quantum mechanics is fundamentally *contextual*: the properties of a quantum system must be considered relative to the context in which they are measured. There is no consistent way of assigning values to all the observables. In [3, 4, 2], contextuality was studied from a sheaf-theoretic point of view, and sheaf cohomology was used to characterise the obstructions to having a consistent global assignment to all the variables. One could say that cohomology detects the *holes* which prevent there being a consistent picture of a global *whole*.

Constraint satisfaction is an important algorithmic paradigm which allows the application of structural methods to central questions of complexity theory. The "non-uniform" version CSP(B) for a fixed finite σ-structure B, where σ is a finite relational vocabulary, asks for an instance given by a finite σ-structure A whether there is a σ-homomorphism $A \to B$. The celebrated Feder-Vardi Dichotomy Conjecture [8] asked whether for every B, CSP(B) is either polynomial-time solvable, or NP-complete. This conjecture was recently proved by Bulatov and Zhuk [5, 12].

Recently, Adam Ó Conghaile has pointed out surprisingly close connections between these two, prima facie completely unrelated topics [7], further developed in [1].

- The idea of k-consistency in constraint satisfaction, an approximation method which yields exact results in a wide range of cases, is naturally represented as the coflasquification (dual to the well-known Godement construction [9]) of a sheaf of partial homomorphisms.
- These representations take the same form as the sheaf-theoretic representations of contextuality in [3]. This in turn allows the cohomological criteria for contextuality introduced in [4, 2] to be used to give a computationally efficient refinement of k-consistency.
- The results in [4, 2] can be leveraged to show that this refined version of k-consistency gives exact results for all affine templates, which form one of the main tractable classes for which the standard k-consistency algorithm fails.
- Current work is aimed at determining the exact power of the cohomological refinement of k-consistency.
- The same ideas can be adapted to give a very similar analysis for the widely studied Weisfeiler-Leman equivalences [10], which give polynomial-time approximations to graph and structure isomorphism. Cohomological refinements of these equivalences can then be introduced, and are shown in [7] to defeat various families of counter-examples based on the Cai-Furer-Immerman construction [6], which is paradigmatic in finite model theory.

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