A representation theorem for a system of point-free geometry*

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In Gerla and Gruszczyński (2017) we put forward a system of geometry based on the primitive notions of *region*, *parthood* and *oval*, the last one being a counterpart of the well-known notion of *convex set*. Regions are interpreted as elements of a complete atomless Boolean algebra, parthood as the standard Boolean order, and ovals as elements of a distinguished set of regions.

The notion of *oval*[1] is very expressive from geometrical point of view, and by means of it we can define well-known standard notions. For example:

1. a *half-plane* is any element of the set $\mathbf{H} \subseteq \mathbf{O}^+$ closed for the Boolean complement,[2]
2. a line is a pair $\langle x, y\rangle$ of non-zero ovals that is maximal with respect to the pointwise order in $\mathbf{O}^+ \times \mathbf{O}^+$ (inherited from the standard pointwise order on the product of the algebra), x and y are *sides* of the line,
3. lines L_1 and L_2 are parallel iff they have disjoint sides,
4. a line L *crosses* a region x iff both sides of L overlap x (i.e., have the non-zero meet with x),
5. regions $x_1, \ldots, x_n$ are *aligned* iff there is a line L that crosses them all,
6. an *angle* is any region that is the meet of the sides of non-parallel lines, and a *stripe* is the non-zero meet of two half-planes that are sides of two parallel lines,
7. the *hull* of a region x (in symbols: $\mathrm{hull}(x)$) is the infimum of all ovals that contain x.

With the concepts defined above in (Gerla and Gruszczyński, 2017) we formulated the following axioms for structures $\langle \mathbf{R}, \leq, \mathbf{O}\rangle$:

$$\langle \mathbf{R}, \leq\rangle \text{ is a complete atomless Boolean lattice.} \tag{00}$$

$$\mathbf{O} \text{ is an algebraic closure system in } \langle \mathbf{R}, \leq\rangle \text{ containing } \mathsf{0}. \tag{01}$$

$$\mathbf{O}^+ \text{ is dense in } \langle \mathbf{R}^+, \leq\rangle\,. \tag{02}$$

$$\text{The sides of a line form a partition of } \mathsf{1}\,. \tag{03}$$

$$\text{For any } a, b, c \in \mathbf{O}^+ \text{ which are not aligned there is a line which separates } a \text{ from } \mathrm{hull}(b + c). \tag{04}$$

$$\text{If distinct lines } L_1 \text{ and } L_2 \text{ both cross an oval } a, \text{ then they split } a \text{ into at least three parts.} \tag{05}$$

$$\text{No half-plane is part of any stripe or angle.} \tag{06}$$

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[1] Let $\mathbf{O}$ be the set of all ovals of a given structure.

[2] For any set of regions A, $A^+ := A \setminus \{\mathsf{0}\}$, where $\mathsf{0}$ is the minimal element of the algebra.

The structure composed of regular open subsets of the two-dimensional Cartesian plane with ovals interpreted as its convex regions is a model of (O0)–(O6), therefore the system is consistent.

In (Gerla and Gruszczyński, 2017) we proved that in this system all axioms of a point-free system geometry by Śniatycki (1968) are provable, and thus using the results of the latter paper, by means the notions of *regions*, *part of* and *oval* we can define the standard geometrical concepts of *point* and *betweenness*, and moreover we can prove all the axioms of the betweenness fragment of geometry (i.e., we can capture affine geometry).

The intention of this talk is to present a sketch of the proof of the following representation theorem, which is an extension of the results from (Gerla and Gruszczyński, 2017):

Theorem 1. *For any point-free system of geometry satisfying axioms* (O0)–(O6) *there is a topological space* $\langle \Pi, \mathscr{O} \rangle$ *with a notion of convex set* $\mathbf{C}$*, such that* $\langle \mathbf{R}, \leq \rangle$ *is isomorphic with the algebra of regular open subsets of* Π *via a certain function* f*, and for any region* x: $x \in \mathbf{O}$ *iff* $f(x) \in \mathbf{C}$.

Intuitively, points of the topological space are constructed from geometrical entities as equivalence classes of non-parallel half-planes. This goes along the intuition that a point on a plane can be identified with a pair of intersecting lines. Further, any pair of non-parallel half-planes determines a four-element partition of the unity of the Boolean algebra. Using this we can define an *internal* point α of a region x, by requiring that for every representative P of this point (i.e., any pair of non-parallel half-planes that represents α), all four elements of the partition determined by P meet some oval a that is a part of x (in the sense that they have non-zero boolean meets with a). The idea is that we can take the family of all sets of internal points of ovals as a basis for the topological space from the theorem, and we can take the mapping taking regions to their internal points to establish the representation. In particular, we may prove that for any oval a the set of its internal points is a convex set in the topological space, where a convex set is characterized by means of the betweenness relation in the standard way: as the set that together with any pair α and β of its points contains all points lying between α and β.

References

Gerla, G. and Gruszczyński, R. (2017). Point-free geometry, ovals and half-planes. *Review of Symbolic Logic*, 10(2):237–258.

Śniatycki, A. (1968). *An axiomatics of non-Desarguean geometry based on the half-plane as the primitive notion.* Number LIX in Dissertationes Mathematicae. PWN, Warszawa.