Sums of Kripke frames and locally finite modal logics

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In classical model theory, there is a number of results ("composition theorems") that reduce the theory (first-order, MSO) of a compound structure (e.g., sum or product) to the theories of its components, see, e.g., [Gur85]. In this talk we discuss the composition method in the context of modal logic.

We consider the operation of sum on Kripke frames, where a family of frames-summands is indexed by elements of another frame. In many cases, the modal logic of sums inherits the finite model property and decidability from the modal logic of summands [BR10], [Sha18]. Under a general condition, the satisfiability problem on sums is polynomial space Turing reducible to the satisfiability problem on summands; in particular, for many modal logics decidability in PSpace is an immediate corollary from the semantic characterization of the logic [Sha22].

In this talk we announce the following result: if both the logic of indices and the logic of summands are locally finite, then the logic of sums is also locally finite. We also formulate a sufficient syntactic condition for local finiteness of bimodal logics.

Main result

Fix an $\mathrm{A} < \omega$ for the alphabet of modal operators.

Definition 1. Consider a family $(\mathsf{F}_i)_{i\in I}$ of A-frames $\mathsf{F}_i = (W_i, (R_{i,a})_{a\in\mathrm{A}})$. The *sum* $\sum_{i\in\mathsf{I}} \mathsf{F}_i$ *of the family* $(\mathsf{F}_i)_{i\in I}$ *of* A*-frames over an* A*-frame* $\mathsf{I} = (I, (S_a)_{a\in\mathrm{A}})$ is the A-frame $(\bigsqcup_{i\in I} W_i, (R_a^\Sigma)_{a\in\mathrm{A}})$, where $\bigsqcup_{i\in I} W_i = \bigcup_{i\in I}(\{i\}\times W_i)$ is the disjoint union of sets W_i, and

$$(i,w)R_a^\Sigma(j,v) \quad \text{iff} \quad (i=j \;\&\; wR_{i,a}v) \text{ or } (i\neq j \;\&\; iS_aj).$$

For classes $\mathcal{I}$, $\mathcal{F}$ of A-frames, let $\sum_{\mathcal{I}}\mathcal{F}$ be the class of all sums $\sum_{i\in\mathsf{I}}\mathsf{F}_i$ such that $\mathsf{I}\in\mathcal{I}$ and $\mathsf{F}_i\in\mathcal{F}$ for every i in I.

Modal logics of sums appear in various contexts such as provability logic, complexity and decision problems, completeness problems; see, e.g., [Bek10, Sha08, Bal09, BR10, Sha18, Sha22].

Theorem 1. *Let* $\mathcal{F}$ *and* $\mathcal{I}$ *be classes of* A*-frames. If the modal logics* $\mathrm{Log}(\mathcal{F})$ *and* $\mathrm{Log}(\mathcal{I})$ *are locally finite, then the logic* $\mathrm{Log}(\sum_{\mathcal{I}}\mathcal{F})$ *is locally finite as well.*

The proof is based on the semantic criterion of local finiteness given in [SS16] (Theorem 4.3).

Lexicographic sums

The sum operation given above does not change the signature. In many cases it is convenient to characterize a polymodal logic via the following variant of the sum operation.

Definition 2. Let $\mathsf{I} = (I,S)$ be a unimodal frame, $(\mathsf{F}_i)_{i\in I}$ a family of A-frames, $\mathsf{F}_i = (W_i, (R_{i,a})_{a\in\mathrm{A}})$. The *lexicographic sum* $\overset{\text{lex}}{\sum_{\mathsf{I}}}\mathsf{F}_i$ is the $(1+\mathrm{A})$-frame $\left(\bigsqcup_{i\in I} W_i, S^{\text{lex}}, (R_a)_{a<N}\right)$, where

$$(i,w)S^{\text{lex}}(j,u) \quad \text{iff} \quad iSj,$$
$$(i,w)R_a(j,u) \quad \text{iff} \quad i=j \;\&\; wR_{i,a}u.$$

For a class $\mathcal{F}$ of A-frames and a class $\mathcal{I}$ of 1-frames, $\sum\limits^{\text{lex}}_{\mathcal{I}}\mathcal{F}$ denotes the class of all sums $\sum\limits^{\text{lex}}_{\mathsf{I}}\mathsf{F}_i$, where $\mathsf{I} \in \mathcal{I}$ and all F_i are in $\mathcal{F}$. For a unimodal L_1, let $\sum\limits^{\text{lex}}_{L_1} L_2$ be the logic of the class $\sum\limits^{\text{lex}}_{\text{Frames}\, L_1} \text{Frames}\, L_2$.

In the case when all summands are equal, this operation is the *lexicographic product*; lexicographic products of modal logics were introduced in [Bal09].

Theorem 2. *Let L_1 be a unimodal logic, L_2 be an* A*-modal logic. If L_1 and L_2 are locally finite, then the logic $\sum\limits^{\text{lex}}_{L_1} L_2$ is locally finite as well.*

This theorem is an easy corollary of Theorem 1.

Consider the 2-modal formulas $\alpha = \Diamond_1\Diamond_0 p \to \Diamond_0 p$, $\beta = \Diamond_0\Diamond_1 p \to \Diamond_0 p$, $\gamma = \Diamond_0 p \to \Box_1\Diamond_0 p$. One can see that these formulas are valid in every lexicographic sum $\sum\limits^{\text{lex}}_{\mathsf{I}}\mathsf{F}_i$ of 1-frames F_i. In many cases, α, β, γ provide a complete axiomatization of $\sum\limits^{\text{lex}}_{L_1} L_2$, that is we have

$$\sum\limits^{\text{lex}}_{L_1} L_2 = L_1 * L_2 + \{\alpha, \beta, \gamma\}, \tag{1}$$

where $L_1 * L_2$ denotes the *fusion* of unimodal L_1 and L_2, $L + \Psi$ denotes the smallest normal logic containing $L \cup \Psi$. In particular, (1) holds for the logic $\sum\limits^{\text{lex}}_{\text{GL}}\text{GL}$ [Bek10] (where GL is the Gödel-Löb logic) and for $\sum\limits^{\text{lex}}_{\text{S4}}\text{S4}$ [Bal09].

Theorem 3. *Let L_1 and L_2 be locally finite canonical unimodal logics. If the class* Frames L_1 *is definable in first-order language without equality, then the logic $L_1 * L_2 + \{\alpha, \beta, \gamma\}$ is locally finite.*

The proof follows from the fact that under the condition of the theorem, (1) holds for L_1 and L_2.

References

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