One-Sorted Program Algebras

Igor Sedlár* and Johann J. Wannenburg

Czech Academy of Sciences, Institute of Computer Science
{sedlar,wannenburg}@cs.cas.cz

A *Kleene algebra* [3] is a structure $(K, \cdot, +, ^*, 1, 0)$ where $(K, +, \cdot, 1, 0)$ is an idempotent semiring and $^* : K \to K$, the Kleene star operation, satisfies

$$1 + x + x^*x^* \leq x^* \tag{1}$$

$$yx \leq x \implies y^*x \leq x \tag{2}$$

$$xy \leq x \implies xy^* \leq x \tag{3}$$

A Kleene algebra is *-*continuous* iff

$$xy^*z = \sum_{n \geq 0} xy^n z\,. \tag{4}$$

Kleene algebra formalizes equational reasoning about regular languages and algebras of binary relations. Kleene algebras with tests [4], KAT, a two-sorted generalization of Kleene algebra containing a Boolean subalgebra of tests, formalizes equational reasoning about while programs.

Kleene algebra with (Boolean) domain KAD [1, 2] provides a one-sorted alternative to KAT. KAD expands KA with two unary operators d and a such that

$$x \leq \mathsf{d}(x)x \tag{5}$$

$$\mathsf{d}(xy) = \mathsf{d}(x\mathsf{d}(y)) \tag{6}$$

$$\mathsf{d}(x) \leq 1 \tag{7}$$

$$\mathsf{d}(0) = 0 \tag{8}$$

$$\mathsf{d}(x + y) = \mathsf{d}(x) + \mathsf{d}(y) \tag{9}$$

$$\mathsf{a}(x) + \mathsf{d}(x) = 1 \tag{10}$$

$$\mathsf{d}(x)\mathsf{a}(x) = 0 \tag{11}$$

A symmetric variant of KAD is *Kleene algebra with (Boolean) codomain*, KAC; its axiomatization results from the axiomatization of KAD by replacing $\mathsf{d}(x)x$ with $x\mathsf{c}(x)$ in the first axiom, $x\mathsf{d}(y)$ with $\mathsf{c}(x)y$ in the second axiom and d with c in the rest. In each Kleene algebra with domain, the test algebra $(\mathsf{d}(K), \cdot, +, \mathsf{a}, 1, 0)$ is a Boolean algebra, and similarly for $\mathsf{c}(K)$ in Kleene algebras with codomain. Consequently, the equational theory of KAT embeds to the equational theory of KAD (and KAC). However, from the viewpoint of Kleene algebra with tests, KAD and KAC have some peculiar features: the test algebra is necessarily the largest Boolean subalgebra of the negative cone (of the underlying Kleene algebra), and not every Kleene algebra expands to a Kleene algebra with domain (or codomain), the culprit being the locality axiom (6).

In the first part of the talk we introduce a generalization of KAD and KAC that avoids their peculiar features while retaining their good properties. *One-sorted Kleene algebra with tests* OneKAT expands Kleene algebra with two unary operations t and t' such that

$$\mathsf{t}(0) = 0 \tag{12}$$

$$\mathsf{t}(1) = 1 \tag{13}$$

$$\mathsf{t}(\mathsf{t}(x) + \mathsf{t}(y)) = \mathsf{t}(x) + \mathsf{t}(y) \tag{14}$$

$$\mathsf{t}(\mathsf{t}(x)\mathsf{t}(y)) = \mathsf{t}(x)\,\mathsf{t}(y) \tag{15}$$

$$\mathsf{t}(x)\mathsf{t}(x) = \mathsf{t}(x) \tag{16}$$

$$\mathsf{t}(x) \leq 1 \tag{17}$$

$$1 \leq \mathsf{t}'(\mathsf{t}(x)) + \mathsf{t}(x) \tag{18}$$

$$\mathsf{t}'(\mathsf{t}(x))\,\mathsf{t}(x) \leq 0 \tag{19}$$

$$\mathsf{t}'(\mathsf{t}(x)) = \mathsf{t}(\mathsf{t}'(\mathsf{t}(x))) \tag{20}$$

*Speaker.

We will show that the test algebra $(\mathsf{t}(K), \cdot, +, \mathsf{t}', 1, 0)$ of each OneKAT algebra is a Boolean algebra and that the equational theory of KAT embeds into the equational theory of OneKAT; moreover, every Kleene algebra expands to a OneKAT algebra and the test algebra of a OneKAT algebra is not necessarily the maximal Boolean subalgebra of the negative cone of the underlying Kleene algebra. We will also show that adding "back" some KAD axioms—such as additivity (9), left preserver (5) or sublocality $\mathsf{t}(xy) \leq \mathsf{t}(x\mathsf{t}(y))$—does not change this.

In the second part of the talk, we consider a particular extension of OneKAT called S-type OneKAT algebras, SKAT. An S-type OneKAT algebra is $(K, \cdot, +, \rightarrow, \hookrightarrow, {}^*, 1, 0, \mathsf{t}, \mathsf{e})$ where $(K, \cdot, +, \rightarrow, \hookrightarrow, {}^*, 1, 0)$ is a residuated Kleene algebra, that is

$$y \leq x \hookrightarrow z \iff xy \leq z \iff x \leq y \rightarrow z\,, \tag{21}$$

and t, e are unary operators satisfying the following:

$$\mathsf{t}(\mathsf{t}(x)\mathsf{t}(y)) = \mathsf{t}(x)\,\mathsf{t}(y) \tag{15}$$
$$\mathsf{t}(x) \leq 1 \tag{17}$$
$$\mathsf{t}(x+y) = \mathsf{t}(x) + \mathsf{t}(y) \tag{22}$$
$$\mathsf{e}(x) \leq \mathsf{e}(x+y) \tag{23}$$
$$x \leq \mathsf{e}(\mathsf{t}(x)) \tag{24}$$
$$\mathsf{t}(\mathsf{e}(x)) \leq x \tag{25}$$
$$x \leq x\mathsf{t}(x) \tag{26}$$
$$\mathsf{t}(xy) \leq \mathsf{t}(\mathsf{t}(x)y) \tag{27}$$
$$\mathsf{t}(x \rightarrow y) \leq x \rightarrow x\mathsf{t}(y) \tag{28}$$
$$1 \leq \mathsf{t}(\mathsf{t}(x) \rightarrow 0) + \mathsf{t}(x) \tag{29}$$

We define $\mathsf{t}'(x) = \mathsf{t}(\mathsf{t}(x) \rightarrow 0)$. The operators t and e form a Galois connection. SKAT is an expansion of KAC with residuals and e, and of Pratt's action algebras [6] with t and e. SKAT is a variety since both (2–3) and (21) can be replaced by equations already in action algebras. The equational replacement of (2–3) is made possible by the presence of the residuals $\rightarrow$ and $\hookrightarrow$, together with the equational axioms of "pure induction", $(x \rightarrow x)^* \leq x \rightarrow x$ and $(x \hookrightarrow x)^* \leq x \hookrightarrow x$; see [6].

We show that the three-sorted substructural logic of partial correctness S, introduced by Kozen and Tiuryn in [5], embeds into the equational theory of *-continuous SKAT algebras.

References

[1] J. Desharnais and G. Struth. Modal semirings revisited. In *International Conference on Mathematics of Program Construction*, pages 360–387. Springer, 2008.

[2] J. Desharnais and G. Struth. Internal axioms for domain semirings. *Science of Computer Programming*, 76(3):181–203, 2011. Special issue on the Mathematics of Program Construction (MPC 2008).

[3] D. Kozen. A completeness theorem for Kleene algebras and the algebra of regular events. *Information and Computation*, 110(2):366 – 390, 1994.

[4] D. Kozen. Kleene algebra with tests. *ACM Trans. Program. Lang. Syst.*, 19(3):427–443, May 1997.

[5] D. Kozen and J. Tiuryn. Substructural logic and partial correctness. *ACM Trans. Computational Logic*, 4(3):355–378, July 2003.

[6] V. Pratt. Action logic and pure induction. In *Logics in AI. European workshop JELIA '90, Amsterdam, the Netherlands, September 10-14, 1990. Proceedings*, pages 97–120. Berlin etc.: Springer-Verlag, 1991.