Modal logic over semi-primal algebras

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In his generalized 'Boolean' theory of universal algebras [4] Foster introduced primal algebras. Generalizing the two-element Boolean algebra $\mathbf{2}$, an algebra $\mathbf{L}$ is *primal* if every operation on its carrier set L is term-definable. During the second half of the 20th century, various weakenings of this property have been studied [9]. Since the algebras thus arising are still 'close to $\mathbf{2}$', it is reasonable to consider them as algebras of truth-values for many-valued logic. In the talk we focus on *semi-primality* [5].

1 Definition. A finite algebra $\mathbf{L}$ is *semi-primal* if every operation $f\colon L^n \to L$ which preserves subalgebras[1] is term-definable in $\mathbf{L}$.

In a slogan, semi-primal algebras are like primal algebras that allow proper subalgebras. Prominent examples from logic are finite Łukasiewicz chains or finite Łukasiewicz-Moisil chains. The framework of our talk is the following.

2 Assumption. Let $\mathbf{L}$ be a semi-primal algebra with underlying bounded lattice and let $\mathcal{A} = \mathbb{HSP}(\mathbf{L})$ be the variety it generates.

Abstractly, $\mathbf{2}$-valued coalgebraic modal logic for an endofunctor $\mathsf{T}\colon \mathsf{Set} \to \mathsf{Set}$ is summarized in the following picture based on Stone duality after 'forgetting topology':

$$\mathsf{T} \circlearrowright \mathsf{Set} \rightleftarrows \mathsf{BA} \circlearrowleft \mathsf{A} \tag{1}$$

For example, if $\mathsf{T} = \mathcal{P}$ is the covariant powerset functor, then the coalgebras $\mathsf{Coalg}(\mathcal{P})$ correspond to Kripke frames and the algebras $\mathsf{Alg}(\mathsf{A})$ correspond to Boolean algebras with operator.

To relate this to our variety $\mathcal{A}$ we apply the duality for semi-primal varieties due to Keimel and Werner [7] (also see [3]) which asserts that $\mathcal{A}$ is dually equivalent to the category $\mathsf{Stone}_{\mathbf{L}}$ defined as follows

3 Definition. Objects of $\mathsf{Stone}_{\mathbf{L}}$ are of the form $(X, \mathbf{v})$ where $X \in \mathsf{Stone}$ and $\mathbf{v}\colon X \to \mathbb{S}(\mathbf{L})$ is continuous. Morphisms $f\colon (X, \mathbf{v}) \to (Y, \mathbf{w})$ in $\mathsf{Stone}_{\mathbf{L}}$ are continuous maps satisfying $\mathbf{w}(f(x)) \leq \mathbf{v}(x)$.

Let $\mathsf{Set}_{\mathbf{L}}$ be the category obtained from $\mathsf{Stone}_{\mathbf{L}}$ after 'forgetting topology'. There is a canonical way to lift T from diagram (1) to an endofunctor $\mathsf{T}'\colon \mathsf{Set}_{\mathbf{L}} \to \mathsf{Set}_{\mathbf{L}}$. We ultimately aim to describe the modal logic abstractly characterized by

$$\mathsf{T}' \circlearrowright \mathsf{Set}_{\mathbf{L}} \rightleftarrows \mathcal{A} \circlearrowleft \mathsf{A}' \tag{2}$$

This also yields the more commonly investigated case

$$\mathsf{T} \circlearrowright \mathsf{Set} \rightleftarrows \mathcal{A} \circlearrowleft \mathsf{A}' \tag{3}$$

obtained after composing by the forgetful functor $\mathsf{U}\colon \mathsf{Set}_{\mathbf{L}} \to \mathsf{Set}$ and its left adjoint.

[1] If $\mathbf{S}$ is a subalgebra of $\mathbf{L}$ then $a_1 \dots a_n \in S \Rightarrow f(a_1, \cdots a_n) \in S$.

4 Example. In our first example, let $\mathsf{T} = \mathcal{P}$. The coalgebras for the lifted functor $\mathsf{Coalg}(\mathcal{P}')$ correspond to *crisp* $\mathbf{L}$-*frames*. That is, to triples $\mathfrak{F} = (W, R, \mathbf{v})$ where (W, R) is a Kripke frame and $\mathbf{v}\colon W \to \mathbb{S}(\mathbf{L})$ satisfies the compatibility condition

$$wRw' \Rightarrow \mathbf{v}(w') \subseteq \mathbf{v}(w)$$

For the $\mathbf{L}$-*models* over $\mathfrak{F}$ we only allow valuations $Val : W \times \mathsf{Prop} \to L$ which always satisfy

$$Val(w, p) \in \mathbf{v}(w).$$

In this case, diagram (2) is closely related to work by Maruyama [8]: the algebras $\mathsf{Alg}(\mathsf{A}')$ correspond to what is therein called $\mathbb{ISP}_{\mathbb{M}}(\mathbf{L})$. The non-restricted case where all valuations are allowed corresponds to diagram (3) and arises if $\mathbf{v}(w) = \mathbf{L}$ everywhere. Here, in the special case $\mathbf{L} = \mathbf{Ł}_n$ it corresponds to modal extensions of Łukasiewicz many-valued logic as described in [6].

5 Example. For another example, we hint at the case where $\mathsf{T} = \mathcal{L}$ is the covariant functor which generalizes $\mathcal{P}$, that is, it is defined on objects by $\mathcal{L}(X) = L^X$ and assigns to a morphism $f : X \to Y$ the morphism $\mathcal{L}f : L^X \to L^Y$ given by

$$h \mapsto (y \mapsto \bigvee\{h(x) \mid f(x) = y\}).$$

Now in (2) the coalgebras for the lifted endofunctor $\mathsf{Coalg}(\mathcal{L}')$ correspond to the $\mathbf{L}$-*labeled* $\mathbf{L}$-*frames*, that is, $(W, R, \mathbf{v})$ similar to the crisp $\mathbf{L}$-frames except that now the accessibility relation $R\colon W \to L^W$ is many-valued as well. Diagram (3) corresponds again to $\mathbf{L}$-labeled frames without further restrictions. This, in the case $\mathbf{L} = \mathbf{Ł}_n$ corresponds to the frames that have been recently investigated by algebraic means in [2] (see also [1]).

In the talk, we will report about our work in progress on the investigation of the modal logics arising from diagrams (2) and (3) in the general case, and illustrate some examples which arise by specifying to some particular functors T.

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