# An algebraic theory of clones


Antonino Salibra

Université Paris Cité
salibra@unive.it


Clones are sets of finitary operations on a fixed carrier set that contain all projections and are closed under composition. They play an important role in universal algebra, since the set of all term operations of an algebra always forms a clone and in fact every clone is of this form. Clones play another important role in the study of first-order structures. Indeed, the polymorphism clone of a first-order structure, consisting of all finitary functions which preserve the structure, forms a clone. Polymorphism clones carry information about the structures that induce them, and are a powerful tool in their analysis. Clones are also important in theoretical computer science. Many computational problems can be phrased as constraint satisfaction problems (CSP). If we fix a structure $A$, the problem $\mathrm{CSP}(A)$ is the computational problem of deciding whether a given conjunction of atomic formulas over the signature of $A$ is satisfiable in $A$. The seminal discovery in the algebraic approach to CSP is Jeavons's result of [4] that, for a finite structure $A$, the complexity of $\mathrm{CSP}(A)$ is determined by the polymorphism clone of $A$.

A one-sorted algebraic theory of clones has recently been introduced in [2]. Indeed, clone algebras ($\mathsf{CA}$) form a variety of algebras in the universal algebraic sense. A crucial feature of this approach is connected with the role played by variables in free algebras and projections in clones. In clone algebras these are abstracted out, and take the form of a countable infinite system of fundamental elements (nullary operations) $\mathsf{e}_1, \mathsf{e}_2, \dots, \mathsf{e}_n, \dots$ of the algebra. One important consequence of the abstraction of variables and projections is the abstraction of term-for-variable substitution and functional composition in $\mathsf{CA}$s, obtained by introducing an $(n+1)$-ary operator $q_n$ for every $n \geq 0$. Roughly speaking, $q_n(a, b_1, \dots, b_n)$ represents the substitution of $b_i$ for $\mathsf{e}_i$ into $a$ for $1 \leq i \leq n$ (or the composition of $a$ with $b_1, \dots, b_n$ in the first $n$ coordinates of $a$).

In [2] the authors have shown that the finite-dimensional clone algebras generate the variety of clone algebras and are the abstract counterpart of the clones of finitary operations, where the dimension of an element in a clone algebra is an abstraction of the notion of arity. In [2] it was also given an answer to the lattice of equational theories problem proposed by Birkhoff and Maltsev: a lattice is isomorphic to a lattice of equational theories (of finitary algebras) if and only if it is isomorphic to the lattice of all congruences of a finite-dimensional clone algebra.

The most natural $\mathsf{CA}$s, the ones the axioms are intended to characterise, are algebras of functions, called *functional clone algebras* ($\mathsf{FCA}$s). The elements of a $\mathsf{FCA}$ with *value domain* $A$ are infinitary operations, called here t-operations. They are functions $\varphi : \mathsf{a} \to A$, whose domain $\mathsf{a}$, called a *trace on* $A$, is a nonempty subset of $A^\omega$ satisfying the following condition: $\forall r, s \in A^\omega.\ s \in \mathsf{a}$ and $|\{i : s_i \neq r_i\}| < \omega \Rightarrow r \in \mathsf{a}$. A trace $\mathsf{a}$ on $A$ is *complete* if $\mathsf{a} = A^\omega$ and it is *basic* if it is minimal. In this framework the nullary operators are the projections $p_i$, defined by $p_i(s) = s_i$ for every $s \in \mathsf{a}$, and $q_n(\varphi, \psi_1, \dots, \psi_n)$ represents the $n$-ary composition of $\varphi$ with $\psi_1, \dots, \psi_n$, acting on the first $n$ coordinates: $q_n(\varphi, \psi_1, \dots, \psi_n)(s) = \varphi(\psi_1(s), \dots, \psi_n(s), s_{n+1}, s_{n+2}, \dots)$, for every $s \in \mathsf{a}$. Every clone algebra $\mathcal{C}$ of universe $C$ is isomorphic to a $\mathsf{FCA}$ with value domain $C$, whose trace is basic and contains the sequence $(\mathsf{e}_1^{\mathcal{C}}, \mathsf{e}_2^{\mathcal{C}}, \dots, \mathsf{e}_n^{\mathcal{C}}, \dots)$.

The most part of clone algebras are not finite-dimensional. Then it is natural to investigate what are the algebraic structures that correspond to clone algebras in full generality. We have introduced in [5] a new general framework for algebras and clones, called *universal clone* *algebra*. To make a comparison, algebras and clones of finitary operations are to universal algebra what t-algebras and clone algebras are to universal clone algebra. A t-algebra is a tuple $\mathbf{A} = (A, \mathsf{a}, \sigma^{\mathbf{A}})_{\sigma\in\tau}$, where $\mathsf{a}$ is a trace on $A$ and $\sigma^{\mathbf{A}} : \mathsf{a} \to A$ is a t-operation for every $\sigma \in \tau$.

We have two algebraic levels: the lower degree of t-algebras and the higher degree of clone algebras. There are many ways to move between these levels. If $K$ is a class of t-algebras, then $K^{\uparrow}$ is a class of clone algebras. If $H$ is a class of clone algebras, we have two ways to go down: $H^{\downarrow}$ and $H^{\Downarrow}$ are two classes of t-algebras such that $H^{\downarrow} \subseteq H^{\Downarrow}$. After generalising the usual algebraic construction to t-algebras (namely, t-subalgebra, t-product, t-homomorphic image, t-expansion and t-variety), we prove that (1) If $K$ is a t-variety of t-algebras, then $K^{\uparrow}$ is a variety of clone algebras; (2) If $H$ is a variety of clone algebras, then $H^{\downarrow}$ is a t-variety and $H^{\Downarrow}$ is a t-variety closed under t-expansion (*Et-variety*, for short). We provide concrete examples that general results in universal clone algebra, when translated in terms of algebras and clones, give new versions of known theorems in universal algebra.

**Theorem 0.1.** (Birkhoff Theorem for t-algebras) *Let $K$ be a class of t-algebras of the same type. Then the following conditions are equivalent: (1) $K$ is an Et-variety of t-algebras; (2) $K$ is an equational class of t-algebras; (3) $K^{\uparrow}$ is a variety of clone algebras and $K = K^{\uparrow\Downarrow}$.*

**Theorem 0.2.** (Birkhoff Theorem for algebras) *Let $H$ be a class of algebras of the same type and $H^{\star}$ be the class of all t-algebras obtained by gluing together algebras in $H$ (formally defined in [5]). Then the following conditions are equivalent: (1) $H$ is a variety of algebras; (2) $H$ is an equational class of algebras; (3) $H^{\star}$ is an Et-variety of t-algebras; (4) $(H^{\star})^{\uparrow}$ is a variety of clone algebras and $H^{\star} = (H^{\star})^{\uparrow\Downarrow}$.*

The study of topological variants of Birkhoff's theorem was initiated by Bodirsky and Pinsker [1] for locally oligomorphic algebras, and generalised recently by Schneider [6] and Gehrke-Pinsker [3]. These authors provide a Birkhoff-type characterisation of all those members $\mathbf{T}$ of the variety HSP($\mathbf{S}$) generated by a given algebra $\mathbf{S}$, for which the natural homomorphism from Clo$\mathbf{S}$ onto Clo$\mathbf{T}$ is uniformly continuous with respect to the uniformity of pointwise convergence.

If $\mathbf{A}$ is a t-algebra, then $\mathbf{A}^{\uparrow}$ is the term clone algebra over $\mathbf{A}$, the t-algebra analogue of the term clone of an algebra.

**Theorem 0.3.** (Topological Birkhoff for t-algebras) *Let $\mathbf{A}, \mathbf{B}$ be t-algebras of the same type and let $\mathsf{b}$ be the trace of $\mathbf{B}$. Then the following are equivalent: (1) $\mathbf{B}$ is an element of the Et-variety generated by $\mathbf{A}$, and the natural homomorphism from the term clone algebra $\mathbf{A}^{\uparrow}$ onto the term clone algebra $\mathbf{B}^{\uparrow}$ is uniformly continuous. (2) Every t-subalgebra $\mathbf{B}_s$ of $\mathbf{B}$ generated by $s \in \mathsf{b}$ is a t-homomorphic image of a t-subalgebra of a finite t-power of $\mathbf{A}$.*

We remark that the t-subalgebras involved in (2) depend on the trace $\mathsf{b}$. For example, if $s \in \mathsf{b}$ and $|\{s_i : i \in \omega\}| = \omega$, then $\mathbf{B}_s$ is not in general finitely generated. As a corollary, besides the version of topological Birkhoff by Schneider [6] and Gehrke-Pinsker [3], we get new versions of the topological Birkhoff's theorem for algebras depending on the choice of the trace $\mathsf{b}$.

# References


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