Difference–restriction algebras of partial functions: axiomatizations and representations

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Abstract

We investigate the representation and complete representation classes for algebras of partial functions with the signature of relative complement and domain restriction. We provide and prove the correctness of a finite equational axiomatization for the class of algebras representable by partial functions. As a corollary, the same equations axiomatize the algebras representable by injective partial functions. For complete representations, we show that a representation is meet complete if and only if it is join complete. Then we show that the class of completely representable algebras is precisely the class of atomic and representable algebras. As a corollary, the same properties axiomatize the class of algebras completely representable by injective partial functions. The universal-existential-universal axiomatization this yields for these complete representation classes is the simplest possible, in the sense that no existential-universal-existential axiomatization exists.

The study of algebras of partial functions is an active area of research that investigates collections of partial functions and their interrelationships from an algebraic perspective. In pure mathematics, algebras of partial functions arise naturally as structures such as inverse semigroups [10], pseudogroups [7], and skew lattices [8]. In theoretical computer science, they appear in the theories of finite state transducers [3], computable functions [6], deterministic propositional dynamic logics [5], and separation logic [4]. The partial functions are treated as abstract elements that may be combined algebraically using various natural operations. Many different selections of operations have been considered, each leading to a different class/category of abstract algebras (see [9, §3.2] for a guide to the literature). In this talk, we will consider algebras of partial functions for the signature consisting of the two following binary operations:

Relative complement: $f - g := \{(x, y) \mid (x, y) \in f \text{ and } (x, y) \notin g\}$,

Domain restriction: $f \rhd g := \{(x, y) \mid x \in \mathrm{dom}(f) \text{ and } (x, y) \in g\}$.

The choice of this signature was motivated by the following observations:

- we are able to express intersection: $f \cdot g := \{(x, y) \in (x, y) \in f \text{ and } (x, y) \in g\} = f - (f - g)$ (in particular, every algebra of partial functions is naturally equipped with a semilattice structure defined by $f \leq g \iff f \cdot g = f$),
- we are able to compare domains: $\mathrm{dom}(f) \subseteq \mathrm{dom}(g) \iff f \leq g \rhd f$.

[*] Speaker.

Formally, an **algebra of partial functions** of the signature $\{-, \rhd\}$ is a $\{-, \rhd\}$-subalgebra of $(\mathcal{PF}(X), -, \rhd)$, where $\mathcal{PF}(X)$ denotes the set of all partial functions on a set X. **Representable algebras** are those $\{-, \rhd\}$-algebras that are isomorphic to an algebra of partial functions. We will see that the class of representable algebras forms a finitely axiomatizable variety, and exhibit a representation for each such algebra. As a corollary we have that every representable algebra is representable by *injective* partial functions. Inside the class of representable algebras we will then investigate those that admit a **complete representation**, that is, an embedding into an algebra of partial functions turning existing *joins* into *unions* or, equivalently, turning existing *nonempty meets* into *intersections*. In particular, we will see that the completely representable algebras are precisely those algebras that are representable and *atomic*, and that this (universal-existential-universal) axiomatization is the simplest possible, in the sense that no existential-universal-existential axiomatization exists.

This is based on joint work with Brett McLean [1]. In the sequel to this paper, *Difference–restriction algebras of partial functions with operators: discrete duality and completion* [2], we present an adjunction (restricting to a duality) for the category of completely representable algebras and complete homomorphisms, which generalizes the adjunction between atomic Boolean algebras and sets. This is then extended to an adjunction/duality for completely representable algebras equipped with compatibility preserving completely additive operators.

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