Combination of Quantifier-Free Uniform Interpolants using Beth Definability (Abridged Version)

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Combined Uniform Interpolation

We present recent results on *combination of uniform interpolants* [2]. We recall what uniform interpolants are in general. We fix a logic or a theory T and a suitable fragment L (propositional, first-order quantifier-free, etc.) of its language. Given an L-formula $\phi(\underline{x}, \underline{y})$ (here $\underline{x}, \underline{y}$ are the variables occurring in ϕ), a *uniform interpolant (UI)* of ϕ (w.r.t. $\underline{y}$) is a formula $\phi'(\underline{x})$ where only the $\underline{x}$ occur, and satisfying the following two properties: *(i)* $\phi(\underline{x}, \underline{y}) \vdash_T \phi'(\underline{x})$; *(ii)* for any further L-formula $\psi(\underline{x}, \underline{z})$ such that $\phi(\underline{x}, \underline{y}) \vdash_T \psi(\underline{x}, \underline{z})$, we have $\phi'(\underline{x}) \vdash_T \psi(\underline{x}, \underline{z})$. Whenever existing, a uniform interpolant for an entailment like $\phi(\underline{x}, \underline{y}) \vdash_T \psi(\underline{x}, \underline{z})$ is computed *independently* of ψ.

Uniform interpolants were originally studied in non-classical logics, starting from the pioneering work by Pitts [6]. They are a stronger notion than *ordinary Craig interpolants*: indeed, even in the case Craig interpolants exist, uniform interpolants may not exist. Hence, the existence of uniform interpolants is an exceptional phenomenon, but not so infrequent. Since the nineties, they have been extensively studied in a large literature (e.g., [3, 5]).

Recently, the automated reasoning community has developed an increasing interest in uniform interpolants, focusing on the case L is the *quantifier-free* fragment of some first-order theory T: from now on, we restrict our attention to this case. This interest is confirmed, e.g., by Gulwani and Musuvathi in [4], where examples of UI computations were supplied and some algorithms were sketched. The usefulness of uniform interpolants in model checking was first stressed in that work, and then further motivated by data-aware process verification [1].

An important question suggested by model checking concerns the UI transfer to combined theories: *supposing that uniform interpolants exist in theories T_1, T_2, under which conditions do they exist also in the combined theory $T_1 \cup T_2$?* We show that combined uniform interpolants exist in the disjoint signatures convex case under the same hypothesis (i.e., the equality interpolating condition) guaranteeing the transfer of quantifier-free ordinary interpolation. For convex theories we essentially obtain a necessary and sufficient condition. The equality interpolating condition is not sufficient for the non-convex case (see [2] for a counterexample).

Main results. A theory T is *convex* iff for every constraint δ, if $T \vdash \delta \to \bigvee_{i=1}^{n} x_i = y_i$ then $T \vdash \delta \to x_i = y_i$ holds for some $i \in \{1, ..., n\}$. Horn theories are convex, but there exist non-Horn convex theories such as $Th(\mathbb{R}, 0, +, -, =, <)$. We need the following definition:

Definition 1. *A convex universal theory T is* equality interpolating *iff for all variables y_1, y_2 and for every pair of* constraints $\delta_1(\underline{x}, \underline{z}_1, y_1), \delta_2(\underline{x}, \underline{z}_2, y_2)$ *s.t.* $T \vdash \delta_1(\underline{x}, \underline{z}_1, y_1) \wedge \delta_2(\underline{x}, \underline{z}_2, y_2) \to y_1 = y_2$*, there is a term $t(\underline{x})$ s.t.* $T \vdash \delta_1(\underline{x}, \underline{z}_1, y_1) \wedge \delta_2(\underline{x}, \underline{z}_2, y_2) \to y_1 = t(\underline{x}) \wedge y_2 = t(\underline{x})$.

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We recall that a universal theory T has quantifier-free interpolation iff T enjoys amalgamation. In case T is also equality interpolating, a stronger characterization holds:

Fact 1. *The following are equivalent for a convex universal theory T: (i) T is equality interpolating and has quantifier-free interpolation; (ii) T has the strong amalgamation property.*

Consider a primitive formula $\exists \underline{z}\phi(\underline{x}, \underline{z}, y)$: $\exists \underline{z}\, \phi(\underline{x}, \underline{z}, y)$ *implicitly defines* y in T iff the formula $\forall y\, \forall y'\ (\exists \underline{z}\phi(\underline{x}, \underline{z}, y) \land \exists \underline{z}\phi(\underline{x}, \underline{z}, y') \to y = y')$ is T-valid; $\exists \underline{z}\phi(\underline{x}, \underline{z}, y)$ *explicitly defines* y in T iff there is a term $t(\underline{x})$ s.t. the formula $\forall y\ (\exists \underline{z}\phi(\underline{x}, \underline{z}, y) \to y = t(\underline{x}))$ is T-valid. A theory T has the *Beth definability property (for primitive formulae)* iff whenever $\exists \underline{z}\, \phi(\underline{x}, \underline{z}, y)$ implicitly defines the variable y then it also explicitly defines it. It is worth noticing the following result:

Fact 2. *A convex equality interpolating theory T has the Beth definability property.*

Let us fix two theories T_1, T_2 over disjoint signatures Σ_1, Σ_2, satisfying the assumptions of Theorem 1 below. Our problem is to compute a uniform interpolant for $\phi(\underline{x}, \underline{y})$ (w.r.t. $\underline{y}$), where ϕ is a conjunction of $\Sigma_1 \cup \Sigma_2$-literals. In order to design a combined UI algorithm (called ConvexCombCover and shown in detail in [2]), we exploit the equivalence between implicit and explicit definability that is supplied by Beth definability: the algorithm guesses the implicitly definable variables, then eliminates them via explicit definability, and finally uses the component-wise input UI algorithms to eliminate the remaining (not implicitly definable) variables. The identification and the elimination of the implicitly defined variables via explicitly defining terms is essential for the correctness of the combined UI algorithm: when computing a uniform interpolant of $\phi(\underline{x}, \underline{y})$ (w.r.t. $\underline{y}$), the variables $\underline{x}$ are (non-eliminable) parameters, and those variables among the $\underline{y}$ that are implicitly definable *need to be discovered and treated in the same way as the parameters* $\underline{x}$. Only after this, the input UI algorithms can be exploited.

Theorem 1. *Let T_1, T_2 be convex, stably infinite, equality interpolating, universal theories over disjoint signatures admitting uniform interpolants. Then $T_1 \cup T_2$ admits uniform interpolants too. Uniform interpolants in $T_1 \cup T_2$ can be effectively computed using* ConvexCombCover.

The previous theorem shows that the equality interpolating condition is sufficient for transferring uniform interpolants to combinations. In [2], it is also shown that equality interpolating is a necessary condition for obtaining UI transfer, in the sense that it is already required for minimal combinations with signatures adding uninterpreted symbols.

The combination result we obtain is quite strong, as it is a typical 'black box' combination result: it applies not only to theories used in verification (such as the combination of real arithmetics with uninterpreted functions), but also in other contexts.

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