Associativity in Quantum Logic

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Recent years have seen the emergence of a research program aimed at applying the tools and techniques of substructural logic and residuated structures to longstanding problems in quantum logic (see, e.g., [2, 3, 1]). In this context, [4] introduced residuated ortholattices as an abstract algebraic environment for studying residuation in quantum logic. Formally, a *residuated ortholattice* is an algebra $(A, \wedge, \vee, \backslash, \neg, 0, 1)$ such that $(A, \wedge, \vee, \neg, 0, 1)$ is an ortholattice and $\backslash$ is a one-sided residual of the (non-commutative and non-associative) Sasaki product operation given by $x \cdot y = x \wedge (\neg x \vee y)$, i.e., for all $x, y, z \in A$,

$$x \wedge (\neg x \vee y) = x \cdot y \leq z \iff y \leq x\backslash z.$$

In any residuated ortholattice $\mathbf{A}$, one may define a negation operation $\sim x = x\backslash 0$ that satisfies the De Morgan laws. It is shown in [4] that the image of any residuated ortholattice $\mathbf{A}$ under the associated closure operator $x \mapsto \sim\sim x$ is an orthomodular lattice, called the *orthomodular skeleton* of $\mathbf{A}$, and moreover that every orthomodular lattice arises in this fashion. Further, this fact supports a double negation translation interpreting orthomodular lattices in residuated ortholattices, similar to the well known negative translation of classical logic into intuitionistic logic. Accordingly, a better understanding of the structure of residuated ortholattices promises to deepen our understanding of quantum logic, and particularly the relationship between quantum and substructural logics.

Here we offer a preliminary study of the structure of residuated ortholattices, focusing on residuated ortholattices whose orthomodular skeletons are Boolean as a natural entry point. We generalize the classical theory of the commutating elements in orthomodular lattices to residuated ortholattices. Using these technical results, we obtain the following.

Theorem 1. *Sasaki product is associative in a residuated ortholattice $\mathbf{A}$ if and only if the orthomodular skeleton of $\mathbf{A}$ is a Boolean algebra.*

This theorem further highlights the importance of weak associative properties in the study of residuated ortholattices, which was already implicit in [4]. Commensurately, we study various kinds of weak associativity in residuated ortholattices and provide a host of sufficient conditions on x, y, z that guarantee $x \cdot (y \cdot z) = (x \cdot y) \cdot z$ in arbitrary residuated ortholattices. In particular, we obtain the following.

Theorem 2. *Let $\mathbf{A}$ be a residuated ortholattice and let $a, b, c \in A$. If any two of a, b, c have the same image under the map $x \mapsto \sim\sim x$, then $a \cdot (b \cdot c) = (a \cdot b) \cdot c$.*

Weak associative properties have been studied extensively in the context of orthomodular lattices [7, 6, 5], where many results on weak forms of associativity are obtained by checking that they hold in free algebras. Our results apply a fortiori in the the orthomodular case.

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In addition to generalizing most of the known results on weak associativity in orthomodular lattices, we obtain simple equational proofs of these theorems for the first time.

It is well known that Sasaki product is associative in an orthomodular lattice $\mathbf{A}$ if and only if $\mathbf{A}$ is Boolean, as is reflected in Theorem 1 above. This poses the natural question of what equations hold in the $\cdot$ reducts of residuated ortholattices in the presence of associativity. By deploying the structural results obtained so far, we obtain the following.

Theorem 3. *Let $\mathcal{A}$ be the subvariety of residuated ortholattices with associative Sasaki product, and let ε be an equation containing only variables and the operation $\cdot$ of Sasaki product. Then ε holds in $\mathcal{A}$ if and only if ε holds in all left-regular bands.*

In other words, the variety generated by the $\cdot$ reducts of members of $\mathcal{A}$ is precisely the variety of left-regular bands. This observation opens up the possibility of applying the techniques developed in the study of left-regular bands to residuated ortholattices and quantum logic.

References

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