Uniform Lyndon Interpolation for Basic Non-normal Modal and Conditional Logics

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In this talk, via a proof-theoretic method, we show that the non-normal modal logics E, M, EN, MN, MC, K, and their conditional versions, CE, CM, CEN, CMN, CMC, CK, in addition to CKID enjoy the uniform Lyndon interpolation property. This result in particular implies that these logics have uniform interpolation. Although for some of them the latter is known, the fact that they have uniform Lyndon interpolation is new. Also, the proof-theoretic proofs of these facts are new, as well as the constructive way to explicitly compute the interpolants that they provide. On the negative side, we show that the logics CKCEM and CKCEMID enjoy uniform interpolation but not uniform Lyndon interpolation. Moreover, we prove that the non-normal modal logics EC and ECN and their conditional versions, CEC and CECN, do not have Craig interpolation, and whence no uniform (Lyndon) interpolation. This talk is based on a joint work with Amir Akbar Tabatabai and Rosalie Iemhoff. The non-normal modal part was published in WoLLIC 2021 [1] and the extended version with the conditional logics is submitted to its special issue of Journal of Logic and Computation.

In the rest of this abstract, we will discuss the details of the results. Set $\mathcal{L}_\Box = \{\wedge, \vee, \to, \bot, \Box\}$ as the language of modal logics and $\mathcal{L}_\triangleright = \{\wedge, \vee, \to, \bot, \triangleright\}$ as the language of conditional logics. The sets of positive and negative variables of a formula $\varphi \in \mathcal{L}$, denoted respectively by $V^+(\varphi)$ and $V^-(\varphi)$ are defined recursively as expected. Note that $V^+(\varphi \triangleright \psi) = V^-(\varphi) \cup V^+(\psi)$ and $V^-(\varphi \triangleright \psi) = V^+(\varphi) \cup V^-(\psi)$, for $\mathcal{L} = \mathcal{L}_\triangleright$. Define $V(\varphi) = V^+(\varphi) \cup V^-(\varphi)$. Lyndon interpolation (LIP) and Craig interpolation property (CIP) for logics are defined as usual. In the following, we define Uniform Lyndon interpolation (ULIP) and uniform interpolation (UIP) for logics.

Definition 1. A logic L has *ULIP* if for any formula $\varphi \in \mathcal{L}$, atom p, and $\circ \in \{+, -\}$, there are p°-free formulas, $\forall^\circ p\varphi$ and $\exists^\circ p\varphi$, such that $V^\dagger(\exists^\circ p\varphi) \subseteq V^\dagger(\varphi)$, $V^\dagger(\forall^\circ p\varphi) \subseteq V^\dagger(\varphi)$, for any $\dagger \in \{+, -\}$, and $L \vdash \forall^\circ p\varphi \to \varphi$ and $L \vdash \varphi \to \exists^\circ p\varphi$. Moreover, for any p°-free formula ψ if $L \vdash \psi \to \varphi$, then $L \vdash \psi \to \forall^\circ p\varphi$, and $L \vdash \exists^\circ p\varphi \to \psi$. A logic has *UIP* if it has all the mentioned properties, omitting $\circ, \dagger \in \{+, -\}$, everywhere.

The logic E is defined as the smallest set of formulas in $\mathcal{L}_\Box$ containing classical tautologies and closed under modes ponens and the rule $\dfrac{\varphi \leftrightarrow \psi}{\Box\varphi \leftrightarrow \Box\psi}\,(E)$. Other non-normal logics can be defined by adding the following modal axioms to E:

$$\Box(\varphi \wedge \psi) \to \Box\varphi \wedge \Box\psi \quad (\mathrm{M}), \qquad \Box\varphi \wedge \Box\psi \to \Box(\varphi \wedge \psi) \quad (\mathrm{C}), \qquad \Box\top \quad (\mathrm{N}).$$

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We consider the following non-normal modal logics: $\mathsf{EN} = \mathsf{E} + (N), \mathsf{M} = \mathsf{E} + (M), \mathsf{MN} = \mathsf{M} + (N), \mathsf{MC} = \mathsf{M} + (C), \mathsf{K} = \mathsf{MC} + (N), \mathsf{EC} = \mathsf{E} + (C)$, and $\mathsf{ECN} = \mathsf{EC} + (N)$. Similarly, for conditional logics, CE is defined as the smallest set of formulas in $\mathcal{L}_{\triangleright}$ containing classical tautologies and closed under modes ponens and $\dfrac{\varphi_0 \leftrightarrow \varphi_1 \qquad \psi_0 \leftrightarrow \psi_1}{\varphi_0 \triangleright \psi_0 \rightarrow \varphi_1 \triangleright \psi_1}\,(CE)$. The other conditional logics are defined by adding the following conditional axioms to CE:

$$(\varphi \triangleright \psi \wedge \theta) \rightarrow (\varphi \triangleright \psi) \wedge (\varphi \triangleright \theta) \quad \text{(CM)}, \quad (\varphi \triangleright \psi) \wedge (\varphi \triangleright \theta) \rightarrow (\varphi \triangleright \psi \wedge \theta) \quad \text{(CC)},$$

$$\varphi \triangleright \top \quad \text{(CN)}, \quad (\varphi \triangleright \psi) \vee (\varphi \triangleright \neg\psi) \quad \text{(CEM)}, \quad \varphi \triangleright \varphi \quad \text{(ID)}.$$

We consider the following conditional logics: $\mathsf{CEN} = \mathsf{CE} + (CN), \mathsf{CM} = \mathsf{CE} + (CM), \mathsf{CMN} = \mathsf{CM} + (CN), \mathsf{CMC} = \mathsf{CM} + (CC), \mathsf{CK} = \mathsf{CMC} + (CN), \mathsf{CEC} = \mathsf{CE} + (CC), \mathsf{CECN} = \mathsf{CEC} + (CN), \mathsf{CKID} = \mathsf{CK} + (ID), \mathsf{CKCEM} = \mathsf{CK} + (CEM)$, and $\mathsf{CKCEMID} = \mathsf{CKCEM} + (ID)$.

Theorem 2. $(ULIP)$ *The logics* E*,* M*,* MC*,* EN*,* MN*,* K*, their conditional versions* CE*,* CM*,* CMC*,* CEN*,* CMN*,* CK*, and the conditional logic* CKID *have ULIP and hence UIP and LIP.*

(UIP) *The logics* CKCEM *and* CKCEMID *enjoy UIP and hence CIP.*

$(Negative)$ *The logics* EC *and* ECN *and their conditional versions* CEC *and* CECN *do not have CIP. As a consequence, they do not have UIP or ULIP. Moreover, the logics* CKCEM *and* CKCEMID *do not enjoy ULIP.*

Proof sketch. To show our result, we use the sequent calculi for these logics. For modal logics the sequent calculi are defined in [2] and the cut elimination theorem is proved. For conditional logics, we introduce the sequent calculi and prove that the cut rule can be eliminated (the sequent calculi for the logics CK, CKID, CKCEM, and CKCEMID were studied in [3]). To prove ULIP for these logics we extend the notion to sequent calculi. It is easy to see that ULIP for a sequent calculus implies that the corresponding logic has ULIP. Then, using the natural notion of weight on formulas and sequents we can define a well-ordering on the sequents. We use this well-ordering to define the uniform interpolants and prove the desired properties by induction on this well-ordering.

References

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