Free Weak ω-Categories as an Inductive Type

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1 Introduction

Weak ω-categories were first defined by Batanin as algebras for a certain globular operad [1]. Generalising Street's computads, Batanin then defined computads for globular operads as a way to freely generate ω-categories [2]. We provide an alternative, inductive definition of computads for ω-categories, more suitable for computer implementation. This gives an alternative definition of ω-categories, equivalent to the one of Leinster [5], as well as elementary descriptions of the universal cofibrant replacement of Garner [3] for ω-categories, and a new proof of the fact that the category of computads is a presheaf topos.

2 Computads

Our definition of ω-categorical computad is based on a distinguished collection of such objects parametrised by rooted planar trees. We adopt the name *Batanin trees* here to emphasize their interpretation as parametrizing globular pasting diagrams, and we give an inductive definitions of them and of the pasting diagrams they parametrize. We then define by induction on $n \in \mathbb{N}$, the category Comp_n of n-computads and their homomorphisms mutually inductively together with

- a forgetful functor $u_n : \mathsf{Comp}_n \to \mathsf{Comp}_{n-1}$,
- a functor $\mathrm{Cell}_n : \mathsf{Comp}_n \to \mathsf{Set}$ returning the cells of an n-computad
- a functor $\mathrm{Type}_n : \mathsf{Comp}_n \to \mathsf{Set}$ returning pair of parallels cells of an n-computad
- a transformation $\mathrm{ty}_n : \mathrm{Cell}_n \Rightarrow \mathrm{Type}_{n-1}\, u_n$, assigning a type to every cell,
- for every Batanin tree B, an n-computad Pd_n^B and a set $\mathrm{Full}_n(B)$ of n-types of B that "cover" the n-boundary of the B.

An n-computad C consists of an $(n-1)$-computad C_{n-1} and a set of typed variables $V_n^C \to \mathrm{Type}_{n-1}(C_{n-1})$. Homomorphisms $C \to D$ consist of a homomorphism $C_{n-1} \to D_{n-1}$ and a function $V_n^C \to \mathrm{Cell}_n(D)$ compatible with the types. The set $\mathrm{Cell}_n(C)$ is is inductively generated by rules analogous to the term formation rules of the type theory CaTT [4]. Finally, we define a *computad* C to consist of an n-computad C_n for every $n \in \mathbb{N}$ such that $u_{n+1}C_{n+1} = C_n$.

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3 ω-categories

The cells of a computad C form a globular set $\mathrm{Cell}(C)$ and conversely every globular set X gives rise to a be seen as a computad $\mathrm{Free}(X)$ where the source and target of variables are variables themselves. This defines an adjunction $\mathrm{Free} \dashv \mathrm{Cell}$ inducing a finitary monad fc^w on the category of globular sets. We call algebras of this monad *weak ω-categories* and show that computads embed into ω-categories fully faithfully. We eventually show that this notion of ω-category coincides with that of Batanin and Leinster [5]

4 The variable-to-variable subcategory

An important class of homomorphisms of computads are the ones sending variables to variables. Such homomorphisms are closer to the ones defined for Batanin's computads [2] and they form a well-behaved lluf subcategory $\mathsf{Comp}^{\mathrm{var}}$ containing the core of Comp and the image of the functor Free. We then inductively construct familial representations of the functors

$$\mathrm{Cell}^{\mathrm{var}} : \mathsf{Comp}^{\mathrm{var}} \to \mathsf{Glob} \qquad\qquad \mathrm{Type}^{\mathrm{var}} : \mathsf{Comp}^{\mathrm{var}} \to \mathsf{Glob},$$

obtained by restricting the functors of cells and types to the subcategory of variable-to-variable homomorphisms. Using those representations, we show that $\mathsf{Comp}^{\mathrm{var}}$ is a presheaf topos, which famously fails for computads for strict ω-categories. [6, 7].

5 Computadic Replacement

Garner defined a notion of universal cofibrant replacement comonad for a cofibrantly generated weak factorisation system, and gave a description of this comonad for the weak factorisation system on ω-categories generated by the inclusions $\mathbb{S}^{n-1} \to \mathbb{D}^n$ of spheres into disks [3]. We shows that the free ω-category on a computad C is the colimit of the ones free on the n-computads C_n, and that the latter fit in pushout squares of the form

$$\begin{array}{ccc} \coprod_{v\in V_n^C} \mathbb{S}^{n-1} & \rightarrowtail & \coprod_{v\in V_n^C} \mathbb{D}^n \\ \downarrow & & \downarrow \\ C_{n-1} & \rightarrowtail & C_n \end{array}$$

In light of this theorem, we use a construction of Batanin to get a right adjoint $W : \omega\mathsf{Cat} \to \mathsf{Comp}^{\mathrm{var}}$ to the functor sending a computad to the free ω-category it presents. This adjunction defines a comonad Q on $\omega\mathsf{Cat}$ that coincides with the universal cofibrant replacement comonad.

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