Hereditary Structural Completeness over K4

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In deductive systems a rule is said to be *admissible* if the tautologies of the system are closed under its applications and *derivable* if the rule itself holds in the system [5]. Whilst every derivable rule for a system is admissible whether the converse holds, varies between deductive systems. When it does we say the system is *structurally complete*, as one might expect the classical propositional calculus (CPC) is structurally complete, but many non-classical systems including the intuitionistic propositional calculus (IPC) are not [1]. A classical problem is to determine which deductive systems are *structurally complete*. Early investigations suggested it would be possible to precisely characterise the *hereditarily structurally complete* (HSC) systems, those which are not only themselves structurally complete but whose finitary extensions are too. This proved a fruitful question, Citkin [3] produced a characterisation for intermediate logics and Rybakov [6, 7] did so for transitive modal logics. Both these characterisations take a similar form.

Citkin's Theorem An intermediate logic is HSC iff the variety of Heyting algebras associated with it omits five finite algebras [3].

Rybakov's Theorem A transitive modal logic is HSC iff it is not included in the logic of a list of 20 frames [7, pg 274].

Recently, Bezhanishvili and Moraschini [1] gave a new proof of Citkin's theorem. Their approach draws upon both abstract algebraic logic and duality theory. Techniques from abstract algebraic logic allow one to establish that an algebrizable logic is HSC iff its associated variety of algebras is primitive [1, Section 2], that is every all its sub quasi-varieties are in fact varieties. IPC is algebrizable by the variety of Heyting algebra and consequently the task of characterising hereditary structurally complete intermediate logics is equivalent to that of characterising primitive subvarieties of Heyting algebras[1, Section 2]. Results from universal algebra further reduce the problem to centre around the notion of weak projectivity. An algebra A is *weakly projective* in a variety V iff for every $B \in V$ if A is a homomorphic image of B then A is isomorphic to a subalgebra of B.

Lemma 1 Let V be a locally finite variety, that is all its finitely generated members are finite. Then V is primitive iff its finite, non-trivial, finitely subdirectly irreducible (FSI) members are weakly projective in V.

The investigation is further aided through the Esakia duality between Heyting algebras and Esakia spaces[1, Section 3]. This allows the reduced algebraic question to be investigated with topological methods.

Notably a similar framework exists for transitive modal logics; they are algebrizable by the variety of K4-algebras [4] which are linked by Jónnson-Tarski duality to the class of transitive modal spaces. This allows us to do for Rybakov's result what Bezhanishvili and Moraschini did for Citkin's and investigate HSC modal logics through K4-algebras and transitive modal spaces.

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More than simply provide a new proof of Rybakov's theorem, this approach illuminates a mistake in Rybakov's characterisation. The list of frames given by Rybakov is too restrictive and the characterisation of HSC transitive modal logics is revised accordingly.

Theorem 2 The variety generated by the algebra dual to F_3' is primitive, where F_3' is the transitive space $(\{x, y, z\}, \tau, R)$ where $R = \{(x, y), (x, z), (y, z), (z, z)\}$ and τ is the discrete topology.

Revised Theorem A transitive modal logic is HSC iff the variety of K4-algebras associated with it omits the algebras $(F_i)^* : 1 \leq i \leq 17$ and omit the algebra $(G_n)^*$ for some $n \in \omega$.

The proof strategy for the new revised system is the same. However, varieties of K4-algebras are not necessarily locally finite so an alternative to lemma 1 is needed.

Lemma 3 Let V be a variety of K4-algebras. If V is primitive then the finite, non-trivial FSI members of V are weakly projective in V. Moreover, suppose all sub-varieties of V have the finite model property (FMP). Then if the finite, non-trivial FSI members of V are weakly projective in V then V is primitive.

Consequently, the poof strategy for the revised theorem has four components. The first is to establish the easier direction of the revised theorem.

Lemma 4 Primitive varieties of K4-algebras omit the algebras $(F_i)^* : 1 \leq i \leq 17$ and $(G_n)^*$ for some $n \in \omega$.

The second harder direction is much more involved. A crucial step is to give a precise description of the finitely generated, non-trivial, subdirectly irreducible (SI) members of varieties of K4-algebras omitting the given algebras. This description then drives the proofs of the final two key results.

Lemma 5 Let V be a variety of K4-algebras omitting $(F_i)^* : 1 \leq i \leq 17$ and $(G_n)^*$ for some $n \in \omega$. Then V has the FMP.

Lemma 6 Let V be a variety of K4-algebras omitting $(F_i)^* : 1 \leq i \leq 17$ and $(G_n)^*$ for some $n \in \omega$. Every finite, non-trivial FSI member of V is weakly projective in V.

Combing lemmas 3, 4 and 5 then yields a proof of the new revised theorem.

This work is a summary of a master's thesis undertaken at the Institute for Logic, Language and Computation [2].

References

[1] Bezhanishvili, N. and Moraschini, T. (2020), *Hereditarily structurally complete intermediate logics: Citkin's Theorem via Esakia duality*, https://staff.fnwi.uva.nl/n.bezhanishvili/Papers/IPC-HSC.pdf.

[2] Carr, J. (2022), *Hereditary Structural Completeness over K4: Rybakov's Theorem Revisited*, https:msclogic.illc.uva.nl/theses/recent/.

[3] Citkin, A. (1978), *On structurally complete superintuitionistic logics*, Soviet Mathematics Doklady.

[4] Font, J. M. (2016), *Abstract algebraic logic: An introductory textbook*, College Publications.

[5] Raftery, J. (2016) *Admissible Rules and the Leibniz Hierarchy*, Duke University Press.

[6] Rybakov, V.V. (1995) *Hereditarily Structurally Complete Modal Logics*, Association for Symbolic Logic.

[7] Rybakov, V.V. (1997) *Admissibility of Logical Inference Rules*, Elsevier.