The coordinatization of the spectra of ℓ-groups

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Let G be a lattice ordered Abelian group (henceforth, just ℓ-group), the set of prime ideals (=prime convex ℓ-subgroups) of G, endowed with the hull-kernel topology is called the *spectrum* of G and is denoted $\mathtt{Spec}\, G$. Spectra of ℓ-groups have received much attention in the past. It is know that they are *generalised spectral spaces* —i.e. T_0, sober, and with a basis of compact open sets— with the additional property of being *completely normal* —i.e., if x and y are in the closure of a point z, then either x is in the closure of y or y is in the closure of x.

Recently in [4] it was proved that the above properties characterise the second-countable spectra of ℓ-groups; in the same paper it was proved that there cannot be any first order axiomatisation of the distributive lattices that are dual to spectra of ℓ-groups.

It was also observed that spectra of ℓ-groups do not retain enough information to characterise the ℓ-group they come form; e.g., $\mathbb{Z}$ and $\mathbb{R}$ have the same spectrum (a single point). Here we propose a way to attach further information on $\mathtt{Spec}$ so to be able to reconstruct the original ℓ-group, up to isomorphism. More specifically, we provide a way to *coordinatize* $\mathtt{Spec}$. Indeed, although $\mathtt{Spec}$ cannot always be embedded in some power $\mathbb{R}^k$, because of its topological structure, we show that a coordinatization is possible if one enlarges the set of coordinates to an *ultrapower* of $\mathbb{R}$. This is due to the following result.

Theorem 1. *For every cardinal α there exists an ultrapower of $\mathbb{R}$ on an α-regular ultrafilter, in which all linearly ordered groups of cardinality smaller than α embed.*

Since quotients over prime ideals are linearly ordered, one obtains the wanted embedding.

Let $\mathcal{U}$ be an arbitrary ultrapower of $\mathbb{R}$. Any power $\mathcal{U}^k$ can be endowed with a Zariski-like topology by taking as a basis of closed sets:

$$\mathbb{V}(t(x)) \coloneqq \{u \in \mathcal{U}^k \mid \mathcal{U} \models t(u) = 0\},$$

with $t(x)$ ranging among k-ary terms in the language of ℓ-groups. It is easy to see that this topology is not T_0, however it is sober and has a basis of compact open sets. Moreover it is compact only because the origin O belongs to all closed sets. For any ultrapower $\mathcal{U}$ let $\overline{\mathcal{U}^k}$ be the largest T_0 quotient of $\mathcal{U}^k \setminus \{O\}$.

Theorem 2. *For any k, the space $\overline{\mathcal{U}^k}$ is a* generalized spectral space*. Moreover for any ℓ-group G, there exist a cardinal k and an ultrapower $\mathcal{U}$ of $\mathbb{R}$, such that $\mathtt{Spec}\, G$ is homeomorphic to a closed subspace of $\overline{\mathcal{U}^k}$.*

This result has two consequences. The first is that $\mathtt{Spec}$ induces a duality on ℓ-groups — more details about this are contained in another abstract submitted by the same authors. The second consequence is a characterisation of spectra of ℓ-groups as closed subspaces of some $\overline{\mathcal{U}^k}$.

Theorem 3. *Let X be any generalised spectral space. The space X is the spectrum of some ℓ-group if and only if X is, up to an iso, a closed subspace of $\overline{\mathcal{U}^k}$ for some ultrapower $\mathcal{U}$ of $\mathbb{R}$ and some cardinal k.*

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A useful tool to understand the geometry of $\mathcal{U}^n$, with $n \in \mathbb{N}$, is provided by the following decomposition theorem.

Theorem 4 ([2])**.** *If $a \in \mathcal{U}^n$, then $a = \alpha_1 v_1 + \ldots + \alpha_t v_t$ where $\alpha_1, \ldots, \alpha_t \in \mathcal{U}$ are positive, α_{i+1}/α_i is infinitesimal, $v_1, \ldots, v_t$ are orthonormal vectors in $\mathbb{R}^n$, and the decomposition is unique.*

An important contribution to the study of the space of prime ideals of an ℓ-group is [3] (see also [1] for the case with strong unit). There, prime ideals of finitely generated free ℓ-groups are characterised as the sets of piecewise (homogeneous) linear functions with integer coefficients that vanish on a *cone* determined by a tuple of vectors. Using an adaptation of Theorem 4 we are able to connect our results with the above description and provide a more intuitive version using concepts of non-standard analysis. Indeed, we are able to associate to any non-standard point a tuple of orthonormal vectors in $\mathbb{R}^n$, called *index*. This induces a correspondence between prime ideals and sets of indexes. For every prime ideal $\mathfrak{p}$, the set associated to $\mathfrak{p}$ in this way turns out to be the set of all indexes of all non-standard points that are images of $\mathfrak{p}$ under all possible embeddings in Theorem 2, or equivalently, the closure of any image of $\mathfrak{p}$ under these embeddings.

References

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