A Gödel-type translation for non-distributive logics

Andrea De Domenico, Krishna Manoorkar, Alessandra Palmigiano, Mattia Panettiere, and Apostolos Tzimoulis

Vrije Universiteit, Amsterdam

By "non-distributive logics" (aka LE-logics), we understand all logics the algebraics semantics of which is given by varieties of normal lattices-expansions. In [4], a type of (Kripke-style) relational semantics for LE-logics was discussed, which is based on reflexive directed graphs (i.e. tuples (Z, E) such that Z is a set and $E \subseteq Z \times Z$ is reflexive). In the same paper, it was suggested that graph-based semantics supports a conceptual interpretation of LE-logics as *hyperconstructive* logics of evidential reasoning. The proposed talk, which is part of a research program on extending results from intuitionistic logics to LE-logics, reports on a work in progress aimed at extending the Gödel-McKinsey-Tarski (GMT) translation and related results to LE-logics.

As discussed in [5], the semantic underpinning of the GMT translation [6, 9] of intuitionistic logic into the classical normal modal logic S4 is the observation that partial orders $\mathbb{F} = (W, \leq)$ serve as "Kripke frames" for both logics. The difference lies in how the *complex algebra* is defined in each case: when $\mathbb{F}$ is understood as an intuitionistic frame, then the complex algebra of $\mathbb{F}$ is defined as $\mathcal{P}^{\uparrow}(W)$, i.e. the perfect Heyting algebra of the upward-closed subsets of W; when $\mathbb{F}$ is understood as an S4-frame, then the complex algebra of $\mathbb{F}$ is defined as $(\mathcal{P}(W), \Diamond_{\leq})$, i.e. the perfect BAO of the subsets of W with $\Diamond_{\leq}$ satisfying the axioms corresponding to reflexivity and transitivity.

On the algebraic side, for any S4 modal algebra $\mu = (B, \Box)$, the corresponding Heyting algebra is defined by $\mathbb{H}_\mu = (H_\mu, \wedge, \vee, \to)$, where $H_\mu = \{\Box a, a \in B\}$ and $a \to b = \Box(\neg a \vee b)$. On the other hand, given a Heyting algebra $\mathbb{H} = (H, \wedge_H, \vee_H, \to_H)$, its corresponding S4 modal algebra is given by $\mu_{\mathbb{H}} = (B(\mathbb{H}), \Box)$, where $B(\mathbb{H})$ is the free Boolean extension of $\mathbb{H}$ and for any $\alpha = \bigwedge_{i=1}^{m}(\neg c_i \vee d_i)$ for some $c_i, d_i \in H$, we set $\Box(\alpha) = \bigwedge_{i=1}^{m}(c_i \to_H d_i)$. The maps $\phi : \mu \to \mathbb{H}_\mu$, and $\psi : \mathbb{H} \to \mu_{\mathbb{H}}$, provide a starting point for comparing the varieties of S4 modal algebra and Heyting algebras, which has given rise to several transfer theorems and the Blok-Esakia theorem [3, 2, 8, 10, 1].

Recently, in [7], a GMT-type translation between sorted modal logic and lattice logic was explored based on the polarity-based semantics for LE-logics. In our presentation, reflexive graphs $\mathbb{X} = (Z, E)$ serve as relational frames both for propositional lattice logic and for classical normal modal logic T. The difference again lies in how the *complex algebra* is defined: when $\mathbb{X}$ is understood as a lattice logic frame (i.e. a graph-based frame), then the complex algebra $\mathbb{A}$ of $\mathbb{X}$ is defined as the concept lattice of the polarity $\mathbb{P}_{\mathbb{X}} = (Z_A, Z_X, I_{E^c})$; when understood as an S4-frame, the complex algebra $\mathbb{B}$ of $\mathbb{X}$ is defined as $(\mathcal{P}(W), \Diamond_E, \blacklozenge_E)$, i.e. the perfect BAO of the subsets of W with $\Diamond_E, \blacklozenge_E$ satisfying the axiom corresponding to reflexivity and the adjunction related properties between them. We define the GMT-translation $\tau = (\tau_1, \tau_2)$, where $\tau_1, \tau_2 : \mathcal{L}_{\mathrm{LL}} \to \mathcal{L}_{\mathrm{T}}$ by the following recursion:

$$
\begin{array}{rclrcl}
\tau_1(p) & = & \rhd\blacktriangleright p & \tau_2(p) & = & \neg\blacktriangleright p\\
\tau_1(\bot) & = & \rhd\blacktriangleright\bot & \tau_2(\bot) & = & \bot\\
\tau_1(\top) & = & \top & \tau_2(\top) & = & \neg\blacktriangleright\top\\
\tau_1(\phi\wedge\psi) & = & \tau_1(\phi)\wedge\tau_1(\psi) & \tau_2(\phi\vee\psi) & = & \tau_2(\phi)\vee\tau_2(\psi)\\
\tau_1(\phi\vee\psi) & = & \rhd(\blacktriangleright\tau_1(\phi)\wedge\blacktriangleright\tau_1(\psi)) & \tau_2(\phi\wedge\psi) & = & \neg\blacktriangleright(\rhd\neg\tau_2(\phi)\wedge\rhd\neg\tau_2(\psi)).
\end{array}
$$

Where $\blacktriangleright := \neg\blacklozenge_E$ and $\rhd := \neg\Diamond_E$. Unlike the intuitionistic case, two maps τ_1, τ_2, are needed to capture the satisfaction and co-satisfaction relation of LE-logics:

Proposition 1. *For every $\mathcal{L}_{\mathrm{LL}}$-formula ϕ, and every reflexive graph $\mathbb{X} = (Z, E)$,*

$$\begin{array}{lll} \mathbb{X} \Vdash \phi & \textit{iff} & \mathbb{X} \Vdash^* \tau_1(\phi), \\ \mathbb{X} \succ \phi & \textit{iff} & \mathbb{X} \nVdash^* \tau_2(\phi). \end{array}$$

Similary to the intuitionistic case that admissible valuations are restricted to upwards-closed sets, in the LE-logic case the admissible valuations are restricted to Galois-stable sets of the polarity (Z_A, Z_X, I_{E^c}).

On the algebraic side, for any reflexive tense modal algebra $\mathbb{A}$, we define the corresponding lattices $\rho_1(\mathbb{A}) = \mathbb{L}_1 = (\mathrm{L}_1, \vee_1, \wedge_1, \top_1, \bot_1)$, $\rho_2(\mathbb{A}) = \mathbb{L}_2 = (\mathrm{L}_2, \vee_2, \wedge_2, \top_2, \bot_2)$ as follows:

For any $a, b \in \mathbb{A}$,

$$\begin{array}{lll} \mathrm{L}_1 & = & \{\rhd\blacktriangleright a \mid a \in \mathbb{A}\}, \\ \mathrm{L}_2 & = & \{\neg\blacktriangleright a \mid a \in \mathbb{A}\}, \end{array}$$

$$\begin{array}{lllllll} \rhd\blacktriangleright a \vee_1 \rhd\blacktriangleright b & = & \rhd(\blacktriangleright\rhd\blacktriangleright a \wedge \blacktriangleright\rhd\blacktriangleright b) & \quad & \rhd\blacktriangleright a \wedge_1 \rhd\blacktriangleright b & = & \rhd\blacktriangleright a \wedge \rhd\blacktriangleright b, \\ \neg\blacktriangleright a \vee_2 \neg\blacktriangleright b & = & \neg\blacktriangleright a \vee \neg\blacktriangleright b & \quad & \neg\blacktriangleright a \wedge_2 \neg\blacktriangleright b & = & \neg\blacktriangleright(\rhd\blacktriangleright a \wedge \rhd\blacktriangleright b). \end{array}$$

Proposition 2. *For any reflexive tense modal algebra $\mathbb{A}$, $\rho_1(\mathbb{A}) \cong \rho_2(\mathbb{A})$.*

However, unlike intuitionistic logic, given a lattice $\mathbb{L}$, there does not exist a free reflexive tense modal algebra expanding $\mathbb{L}$. In this presentation, we will explain why a free object doesn't exist by defining two minimal distinct tense modal algebras given $\mathbb{L}$, we will discuss the similarities and differences with the intuitionistic case and the comparison of the lattices of corresponding varieties. Finally, we will report on ongoing work on the generalization of the Blok-Esakia theorem and different translation theorems for the GMT translation between the reflexive tense modal logic and the general lattice logic.

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