The Structure of Totally Ordered Idempotent Residuated Lattices

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Residuated lattices encompass a wide array of prominent algebraic structures, with examples including Boolean algebras, Heyting algebras, MV-algebras, De Morgan monoids, relation algebras, and lattice-ordered groups, among many others. Thanks to their diversity, residuated lattices provide a unified treatment of substructural logics, for which they give the equivalent algebraic semantics, as well as connecting these logics to classical algebra. However, this diversity also presents a challenge to offering a broadly-applicable analysis of their structure. One approach to addressing this challenge centers on residuated lattices whose multiplication operation is idempotent. Such algebras have proven important, on both the algebraic and logical level, as components in decomposition theorems for more general residuated lattices (see, e.g., [8, 4]), and also complement the already extensively-pursued study of cancellative residuated lattices (see, e.g., [2, 1]). Analyzing the structure of broad classes of residuated lattices based on associated idempotent algebras depends on obtaining structural descriptions of idempotent residuated lattices themselves.

This study focuses on the structure of totally ordered idempotent residuated lattices, advancing a line of research represented in, e.g., [7, 6, 3]. The right and left inverse operations $x^r = x\backslash 1$ and $x^\ell = 1/x$, where $\backslash$ and $/$ are the two residuals of the underlying monoid operation, play an important role in our inquiry, and are crucial in our study of congruences and subalgebra generation in idempotent residuated chains. Among other things, the properties of the inverse operations allow us to establish the following.

Theorem 1. *The variety of idempotent semilinear residuated lattices has the congruence extension property.*

Inverses also play a pronounced role in the global structure of idempotent residuated chains. In any idempotent residuated chain, the set of elements that are inverses forms a skeleton, which may be realized as the image of a nucleus. We show that it is possible to reconstruct any given totally ordered idempotent residuated lattice as an ordinal sum indexed by its skeleton through considering the partition induced by this nucleus. Further, we characterize the idempotent residuated chains appearing as skeletons by means of a simple identity, which, in the commutative case, identifies the skeletal idempotent residuated chains as odd Sugihara monoids (see [5, 7]).

We further establish that each totally ordered idempotent residuated lattice is determined by its order and inverse operations, together with the multiplicative identity, and illustrate how the multiplication and division operations may be defined from these ingredients. This analysis supports our introduction of *enhanced monoidal preorders*, enrichments of the monoidal preorders considered in [7], and allows us to establish the following result.

*Speaker.

Theorem 2. *Totally ordered idempotent residuated lattices are definitionally equivalent to enhanced monoidal preorders.*

Enhanced monoidal preorders, together with a closely-related graphical presentation of the action of inverses that we call *flow diagrams*, prove a powerful tool for solving problems regarding idempotent residuated chains. We deploy this technology to locate properties causing the failure of the amalgamation property for idempotent residuated chains, which is known to hold under the additional assumption of commutativity. Having pinpointed features that cause amalgamation to fail in the general case, we identify a natural class of idempotent residuated chains for which the amalgamation property holds. In particular, the aforementioned analysis reveals the importance of the derived operation given by $x^\star = x^\ell \wedge x^r$ and suggests consideration of the class of *-*involutive* idempotent residuated chains defined by $x = x^{\star\star}$. Using the structural results mentioned previously together with variants of some results from [10], we establish the following.

Theorem 3. *The class of *-involutive idempotent chains has the strong amalgamation property, and consequently so does the variety of *-involutive idempotent semilinear residuated lattices.*

Because the algebras we consider in this inquiry give the algebraic semantics of certain substructural logics, this work fits into the broader study of metalogical properties of non-classical logics (see e.g. [9]). Via the well known bridge theorems of abstract algebraic logic, we obtain as corollaries of our algebraic results several important metalogical properties for the corresponding logics, including the interpolation property, the Beth definability property, and deduction-detachment theorem.

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