Translational Embeddings via Stable Canonical Rules

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This paper, based on [3], presents a new uniform method for studying modal companions of superintuitionistic (si) deductive systems and related notions, based on the machinery of stable canonical rules developed, e.g., in [1]. Our techniques recover much of the existing theory of modal companions, expand it with new results, and generalize smoothly to rule systems admitting filtrations in richer signatures.

A *si-rule* (*modal rule*) is a pair Γ/Δ with Γ, Δ finite sets of si (modal) formulae. *Si-* and *normal modal rule systems* (defined in [1]) are sets of si- or modal rules axiomatizing universal classes of Heyting algebras and modal algebras respectively, the way si-logics and normal modal logics axiomatize varieties of Heyting and modal algebras. Let $\mathbf{Ext}(\mathtt{IPC})$ and $\mathbf{NExt}(\mathtt{K})$ be the lattices of si- and normal modal logics respectively. For each $\mathtt{L} \in \mathbf{Ext}(\mathtt{IPC})$ there is a least si-rule system $\mathtt{L}_\mathtt{R}$ containing $\varnothing/\varphi$ for each $\varphi \in \mathtt{L}$, and similarly for normal modal logics. Thus the maps $\mathtt{L} \mapsto \mathtt{L}_\mathtt{R}$ and $\mathtt{M} \mapsto \mathtt{M}_\mathtt{R}$ are embeddings of $\mathbf{Ext}(\mathtt{IPC})$ and $\mathbf{NExt}(\mathtt{K})$ into the lattices of si-rule systems $\mathbf{Ext}(\mathtt{IPC}_\mathtt{R})$ and of normal modal rule systems $\mathbf{NExt}(\mathtt{K}_\mathtt{R})$ respectively.

The *Gödel translation* $T(\varphi)$ of a si-formula φ is obtained by prefixing every subformula of φ with $\Box$. Lift the Gödel translation to rules by setting $T(\Gamma/\Delta) := T[\Gamma]/T[\Delta]$. For $\mathtt{L} \in \mathbf{Ext}(\mathtt{IPC})$, set $\tau(\mathtt{L}) := \mathtt{S4} \oplus \{T(\varphi) : \varphi \in \mathtt{L}\}$ and $\sigma(\mathtt{L}) := \mathtt{Grz} \oplus \tau(\mathtt{L})$, and similarly for si-rule systems. For $\mathtt{M} \in \mathbf{NExt}(\mathtt{S4})$, set $\rho(\mathtt{M}) := \{\varphi : T(\varphi) \in \mathtt{M}\}$, and similarly for normal modal rule systems. A normal modal logic (rule system) $\mathtt{M}$ is a *modal companion* of a si-logic (rule system) $\mathtt{L}$ if $\rho(\mathtt{M}) = \mathtt{L}$.

A map $f : \mathfrak{X} \to \mathfrak{Y}$ between Esakia spaces $\mathfrak{X}, \mathfrak{Y}$ is *stable* if continuous and relation preserving. If $\mathfrak{D} \subseteq \wp(Y)$, a map $f : \mathfrak{X} \to \mathfrak{Y}$ satisfies the *bounded domain condition* (BDC) for $\mathfrak{D}$ when for any $x \in X$ and $\mathfrak{d} \in \mathfrak{D}$, if $\uparrow f(x) \cap \mathfrak{d} \neq \varnothing$ then $f[\uparrow x] \cap \mathfrak{d} \neq \varnothing$, where $\uparrow x := \{y : x \leq y\}$. Analogously, stable maps and the BDC are defined for modal spaces. For every finite Esakia space $\mathfrak{F}$ and any $\mathfrak{D} \subseteq \wp(F)$ there is a *si-stable canonical rule* $\eta(\mathfrak{F}, \mathfrak{D})$ which is refuted in an Esakia space $\mathfrak{X}$ iff there is a stable surjection $f : \mathfrak{X} \to \mathfrak{F}$ satisfying the BDC for $\mathfrak{D}$. Similarly, every finite modal space $\mathfrak{F}$ and any $\mathfrak{D} \subseteq \wp(F)$ induce a *modal stable canonical rule* $\mu(\mathfrak{F}, \mathfrak{D})$ which is refuted in a modal space $\mathfrak{X}$ iff there is a stable surjection $f : \mathfrak{X} \to \mathfrak{F}$ satisfying the BDC for $\mathfrak{D}$ [1]. All si- and normal modal rule systems are axiomatizable by stable canonical rules.

Our first main result is an alternative proof of the following theorem.

Theorem 1. *The following pairs of maps are mutually inverse complete lattice isomorphisms:*

1. $\sigma : \mathbf{Ext}(\mathtt{IPC}_\mathtt{R}) \to \mathbf{NExt}(\mathtt{Grz}_\mathtt{R})$ *and* $\rho : \mathbf{NExt}(\mathtt{Grz}_\mathtt{R}) \to \mathbf{Ext}(\mathtt{IPC}_\mathtt{R})$ *[2].*
2. $\sigma : \mathbf{Ext}(\mathtt{IPC}) \to \mathbf{NExt}(\mathtt{Grz})$ *and* $\rho : \mathbf{NExt}(\mathtt{Grz}) \to \mathbf{Ext}(\mathtt{IPC})$ *[4].*

If $\mathfrak{X}$ is a closure space, its *skeleton* $\rho\mathfrak{X}$ is the Esakia obtained by collapsing clusters in $\mathfrak{X}$ and setting $\{\rho[U] : U \in \mathsf{Clop}(\mathfrak{X})\}$ as a basis, where $\rho : \mathfrak{X} \to \rho\mathfrak{X}$ is the map sending each $x \in \mathfrak{X}$ to its cluster. We let $\sigma\rho\mathfrak{X}$ be $\rho\mathfrak{X}$, viewed as a closure space. Theorem 1 follows from lemma 2 below, which we establish using the refutation conditions of stable canonical rules.

Lemma 2. *Let $\mathfrak{X}$ be a* $\mathtt{Grz}$*-space. Then for every modal rule* Γ/Δ*,* $\mathfrak{X} \models \Gamma/\Delta$ *iff* $\sigma\rho\mathfrak{X} \models \Gamma/\Delta$*.*

Proof sketch. ($\Rightarrow$) is easy. To prove ($\Leftarrow$), we assume wlog that $\Gamma/\Delta = \mu(\mathfrak{F}, \mathfrak{D})$, for $\mathfrak{F}$ a finite closure space. If $\mathfrak{X} \not\models \mu(\mathfrak{F}, \mathcal{D})$, then there is a stable surjection $f : \mathfrak{X} \to \mathfrak{F}$ satisfying the BDC for $\mathfrak{D}$. Let $C = \{x_1, \ldots, x_n\} \subseteq F$ be some cluster. By the properties of $\mathtt{Grz}$-spaces, there are disjoint $U_1, \ldots, U_n \in \mathsf{Clop}(\sigma\rho\mathfrak{X})$ with $\rho[M_i] \subseteq U_i$ and $\bigcup_i U_i = \rho[Z_C]$, where $M_i := max(f^{-1}(x_i))$. Thus for each cluster $C \subseteq F$ we may define a map $g_C : \rho[Z_C] \to C$ by setting $z \mapsto x_i \iff z \in U_i$. We combine these into a map $g : \sigma\rho\mathfrak{X} \to F$ by setting $g(\rho(z)) := g_C(\rho(z))$ if $f(z) \in C$ for some proper cluster C, and $g(\rho(z)) := f(z)$ otherwise. It can be shown that g is a stable surjection satisfying the BDC for $\mathfrak{D}$, which establishes $\sigma\rho\mathfrak{X} \not\models \mu(\mathfrak{F}, \mathfrak{D})$. □

We also axiomatically characterize the modal companion maps via stable canonical rules.

Theorem 3. *Let* $\mathtt{L} \in \mathbf{Ext}(\mathtt{IPC_R})$ *be such that* $\mathtt{L} = \mathtt{IPC_R} \oplus \{\eta(\mathfrak{F}_i, \mathfrak{D}_i) : i \in I\}$. *Then we have:*

1. $\tau\mathtt{L} = \mathtt{S4_R} \oplus \{\mu(\sigma\mathfrak{F}_i, \mathfrak{D}_i) : i \in I\}$
2. $\sigma\mathtt{L} = \mathtt{Grz_R} \oplus \{\mu(\sigma\mathfrak{F}_i, \mathfrak{D}_i) : i \in I\}$.

Theorem 4. *Let* $\mathtt{M} \in \mathbf{NExt}(\mathtt{S4_R})$ *with* $\mathtt{M} = \mathtt{S4_R} \oplus \{\mu(\mathfrak{F}_i, \mathfrak{D}_i) : i \in I\}$, *and let* $\rho\mathfrak{D} := \{\rho[\mathfrak{d}] : \mathfrak{d} \in \mathfrak{D}\}$. *Then we have:*

$$\rho\mathtt{M} = \mathtt{IPC_R} \oplus \{\eta(\rho\mathfrak{F}_i, \rho\mathfrak{D}_i) : \mu(\sigma\rho\mathfrak{F}_i, \rho\mathfrak{D}_i) \in \mathtt{M}\}.$$

Theorem 3 follows from the fact that for all si-stable canonical rules $\eta(\mathfrak{F}, \mathfrak{D})$ we have that $T(\eta(\mathfrak{F}, \mathfrak{D}))$ is equivalent to $\mu(\sigma\mathfrak{F}, \mathfrak{D})$ (*rule translation lemma*). We prove Theorem 4 by showing that for any modal stable canonical rule $\mu(\mathfrak{F}, \mathfrak{D})$ with $\mathfrak{F}$ a preorder and for any closure space $\mathfrak{X}$, if $\mathfrak{X} \not\models \mu(\mathfrak{F}, \mathfrak{D})$ then $\rho\mathfrak{X} \not\models \eta(\rho\mathfrak{F}, \rho\mathfrak{D})$ (*rule collapse lemma*).

Lastly, we generalize the Dummett-Lemmon conjecture [5, Corollary 2] to rule systems.

Theorem 5. *A si-rule system* $\mathtt{L} \in \mathbf{Ext}(\mathtt{IPC_R})$ *is Kripke complete iff* $\tau\mathtt{L}$ *is.*

Proof sketch. ($\Leftarrow$) is easy. To prove ($\Rightarrow$), let $\mathtt{L}$ be Kripke complete. Suppose that $\Gamma/\Delta \notin \tau\mathtt{L}$. Wlog, we assume $\Gamma/\Delta = \mu(\mathfrak{F}, \mathfrak{D})$ for $\mathfrak{F}$ a preorder. By rule collapse lemma, $\eta(\rho\mathfrak{F}, \rho\mathfrak{D}) \notin \mathtt{L}$. Since $\mathtt{L}$ is Kripke complete, there is a si Kripke frame $\mathfrak{Y}$ and a stable surjection $f : \mathfrak{Y} \to \rho\mathfrak{F}$ satisfying the BDC for $\rho\mathfrak{D}$. For every $x \in \rho[F]$ look at $\rho^{-1}(x)$, let $k = |\rho^{-1}(x)|$ and enumerate $\rho^{-1}(x) = \{x_1, \ldots, x_k\}$. Working in $\mathfrak{Y}$, for every $y \in f^{-1}(x)$ replace y with a k-cluster $y_1, \ldots, y_k$ and extend the relation R clusterwise. The result, $\mathfrak{Z}$, is a Kripke frame with $\mathfrak{Z} \models \tau\mathtt{L}$. We identify $\rho\mathfrak{Z} = \mathfrak{Y}$. For every $x \in \rho[F]$ define a map $g_x : f^{-1}(x) \to \rho^{-1}(x)$ by setting $g_x(y_i) = x_i$ ($i \leq k$). Finally, define $g : \mathfrak{Z} \to \mathfrak{F}$ by putting $g = \bigcup_{x \in \rho[F]} g_x$. It can be shown that g is a stable surjection satisfying the BDC for $\mathfrak{D}$, thus establishing $\mathfrak{Z} \not\models \mu(\mathfrak{F}, \mathfrak{D})$. □

Via uniform generalizations of our techniques, we obtain similar results in the settings of bi-superintuitionistic and tense deductive systems, and of deductive systems over the modal intuitionistic logic of provability $\mathtt{KM}$ and classical provability logic $\mathtt{GL}$. For details, consult [3].

References

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