Elementary fibrations and groupoids

Jacopo Emmenegger[1,*] , Fabio Pasquali[1] , and Giuseppe Rosolini[1]

DIMA, Università di Genova, Italy

emmenegger@dima.unige.it pasquali@dima.unige.it rosolini@unige.it

Abstract

The aim of the talk is to illustrate a recent result about the comonadicity of elementary fibrations, and to explain its connections with logical equality.

Lawvere's hyperdoctrines [5, 6] mark the beginning of applications of category theory in logic, and they provide a very clear algebraic tool to work with syntactic theories and their extensions in logic. A ***doctrine*** [7] consists of a family of posets indexed on a category with finite products. More precisely, it is a functor $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathcal{Pos}$ into the category $\mathcal{Pos}$ of posets, such that the base category $\mathcal{C}$ has finite products. The controvariant action $P(f)\colon P(Y) \to P(X)$ induced by an arrow $f\colon X \to Y$ is called ***reindexing along*** f. A (possibly multi-sorted) logical theory T gives rise to a primary doctrine P_T as follows. The base category consists contexts and context morphisms, *i.e.* finite lists of typed variables and finite lists of typed terms. Composition is given by substitution of terms in terms and product is concatenation of contexts. The poset $\mathsf{P}_T(x_1\colon X_1, \dots, x_n\colon X_n)$ is the Lindenbaum-Tarksi algebra of formulas in context $(x_1\colon X_1, \dots, x_n\colon X_n)$. Reindexing along a context morphism $(t_1, \dots, t_n)\colon (x_1\colon X_1, \dots, x_m\colon X_m) \to (y_1\colon Y_1, \dots, y_n\colon Y_n)$ is given by substitution of terms in formulas:

$$\phi \in \mathsf{P}_T(y_1\colon Y_1, \dots, y_n\colon Y_n) \longmapsto \phi[{}^{t_1}/_{y_1}, \dots, {}^{t_n}/_{y_n}] \in \mathsf{P}_T(x_1\colon X_1, \dots, x_m\colon X_m).$$

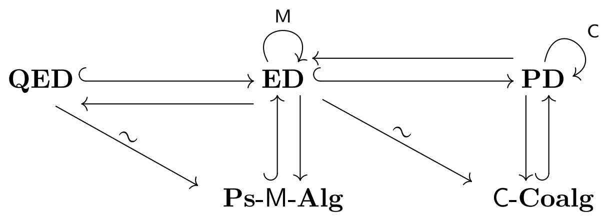
Extending the theory amounts to equip the doctrine with additional structure which, in the spirit of functorial semantics, it is done by requiring certain structural functors to be adjoints. For example, theories with conjunctions correspond to those doctrines, called ***primary***, whose fibres have binary meets which are preserved by reindexing. Adding equality predicates amounts to require that every reindexing along a diagonal $\mathrm{pr}_{1,2,2}\colon Z \times X \to Z \times X \times X$ is right adjoint and satisfies two technical conditions, known as Frobenius Reciprocity and the Beck–Chevalley Condition. These are known as ***elementary doctrines***.

Morphisms between doctrines can be understood as interpretations of a theory into another one, and these can be equipped with a notion of morphism too, giving rise to a 2-category **Doc**. Consider for instance the power set doctrine $\mathcal{P}$ on $\mathcal{Set}$, whose fibre over a set S is the poset of subsets, and reindexing is given by counter-image. Then morphisms into $\mathcal{P}$ are precisely models à la Tarski and, when the theory in question is based on classical first order logic, morphisms between such models are elementary embeddings. Clearly, the model will soundly interpret some logical constant if and only if the morphism into $\mathcal{P}$ preserves the corresponding structure.

The algebraic character of the theory of doctrines makes it a suitable context where to address the question: "What is the theory obtained by (co)freely adding logical structure?", or the closely related question: "How to express additional logical structure in terms of what is already available?". More precisely, in the first case we ask whether a certain forgetful functor is adjoint and, in the second case, whether the adjunction obtained in this way is (co)monadic.

*Speaker.

As a case in point, the forgetful functor from from elementary doctrines to primary ones is comonadic, meaning that elementary doctrines are equivalent to coalgebras for a certain 2-comonad C on **PD** [2]. It is also known that elementary doctrines with quotients are (pseudo) monadic over elementary ones [8] and, interestingly enough, the 2-monad M canonically induced on **ED** turns out to be the one presenting elementary doctrines with quotients as (pseudo) algebras. In particular, the diagram below can be recovered just from the 2-comonad C.



As much as doctrines are well suited to deal with standard "proof-irrelevant" logical systems, they fail to capture the additional complexity present in systems such as type theories, where one wishes to keep track of the different proofs of an entailment $\phi \vdash \psi$. To this aim, it is natural to look at indexed categories instead of indexed posets. These are equivalent, via the Grothendieck construction, to Grothendieck fibrations, which have a more robust theory. Under this equivalence, indexed posets are recovered as those fibrations whose underlying functor is faithful. The description of additional logical structure remains the same as in the faithful case since it is given by adjunctions.

After reviewing the above background and motivation, I will explain how to generalise the comonadicity result to the case of Grothendieck fibrations. An interesting byproduct of the comonadicity of elementary fibrations is that the construction of the 2-comonad is finitary and it shows what the logical intuition supported by the case of doctrines evolves to in the general case: it involves in a crucial way the notion of groupoid.

In fact, the proof is based on a characterisation of elementary fibrations that expose similarities with the structures needed to soundly interpret Martin-Löf's identity type [3]. An instance of such similarities can be found in the fact that Hofmann and Streicher's interpretation of Martin-Löf's identity type in groupoids [4] can be presented as an elementary fibration.

As natural as it is to consider groupoids as higher analogues of equivalence relation, other choices are possible, which I will briefly discuss. For example, the homotopy exact completion of a path category [1] is a coalgebra in the 2-category of fibrations with products. Finally, and if times allow, I plan to discuss work in progress to lift the above diagram to fibrations.

References

[1] B. van den Berg and I. Moerdijk. Exact completion of path categories and algebraic set theory: Part I: Exact completion of path categories. *J. Pure Appl. Algebra*, 222(10):3137 – 3181, 2018.

[2] J. Emmenegger, F. Pasquali, and G. Rosolini. Elementary doctrines as coalgebras. *J. Pure Appl. Algebra*, (224), 2020.

[3] J. Emmenegger, F. Pasquali, and G. Rosolini. A characterisation of elementary fibrations. *Ann. Pure Appl. Logic*, (173), 2022.

[4] M. Hofmann and T. Streicher. The groupoid interpretation of type theory. In *Twenty-five years of constructive type theory* (*Venice, 1995*), volume 36 of *Oxford Logic Guides*, pages 83–111. Oxford Univ. Press, New York, 1998.

[5] F. W. Lawvere. Adjointness in foundations. *Dialectica*, 23:281–296, 1969.

[6] F.W. Lawvere. Equality in hyperdoctrines and comprehension schema as an adjoint functor. In A. Heller, editor, *Proc. New York Symposium on Application of Categorical Algebra*, pages 1–14. Amer.Math.Soc., 1970.

[7] M.E. Maietti and G. Rosolini. Quotient completion for the foundation of constructive mathematics. *Log. Univers.*, 7(3):371–402, 2013.

[8] D. Trotta. *Existential completion and pseudo-distributive laws: an algebraic approach to the completion of doctrines.* PhD thesis, Università di Trento, 2019.