Algebraizable Weak Logics

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Introduction We extend the standard framework of abstract algebraic logic to the setting of logics with limited forms of substitution following recent interest in the algebraic semantics of logics based on team semantics [1, 2, 6]. Failure of closure under uniform substitution precludes us from using the standard definition of algebraizability. We give a modified definition that still preserves the uniqueness of the equivalent algebraic semantics of algebraizable logics. We also show a connection between the modified notion and classical algebraizability of the schematic fragment of a logic.

Algebraizable Logics The notion of algebraizable logic [3, 5] was introduced in the context of abstract algebraic logic in order to give a precise account of the relationship between logics and classes of algebras. For example, Boolean algebras correspond to propositional classical logic, and Heyting algebras correspond to intuitionistic propositional logic. The idea of algebraizability captures the equivalence between a logic and a *unique* class of algebras. A logic can have an algebraic semantics, but still fail to be algebraizable [5].

Weak Logics Given a signature $\mathcal{L}$, a *substitution* is an endomorphism $\sigma : \mathcal{F}m \to \mathcal{F}m$ on the term algebra $\mathcal{F}m$. A *(standard) logic* is a consequence relation $\vdash$ which is closed under *uniform substitution*, i.e. for any substitution σ, $\Gamma \vdash \phi$ entails $\sigma[\Gamma] \vdash \sigma(\phi)$. We are interested in restricting the scope of admissible substitutions. Fix a denumerable set $\mathtt{Var}$ of atomic formulas, let $\mathsf{At}(\mathcal{L})$ denote the set of all substitutions σ such that $\sigma[\mathtt{Var}] \subseteq \mathtt{Var}$.

Definition 1 (Weak Logic)**.** A finitary consequence relation $\vdash$ is a *weak logic* if for all substitutions $\sigma \in \mathsf{At}(\mathcal{L})$, $\Gamma \vdash \phi$ entails $\sigma[\Gamma] \vdash \sigma(\phi)$.

In order to make sense of weak logics from an algebraic perspective, we supplement a standard $\mathcal{L}$-algebra $\mathcal{A}$ with an extra predicate symbol P. We call the resulting structure an *expanded algebra* and refer to the interpretation $P^{\mathcal{A}}$ as $\mathsf{core}(\mathcal{A})$. Intuitively, the core of an expanded algebra captures all the elements that can be substituted for freely.

Definition 2 (Core Semantics)**.** If $\mathbf{K}$ is a class of expanded algebras and $\Theta \cup \{\epsilon \approx \delta\}$ is a set of equations, then $\Theta \vDash^c_K \epsilon \approx \delta \iff$ for all $\mathcal{A} \in \mathbf{K}$, for all $h : \mathcal{F}m \to \mathcal{A}$ such that $h[\mathtt{Var}] \subseteq \mathsf{core}(\mathcal{A})$, if $h(x) = h(y)$ for all $x \approx y \in \Theta$, then $h(\epsilon) = h(\delta)$.

We say that an expanded algebra $\mathcal{A}$ is *core-generated* if $\mathcal{A} = \langle \mathsf{core}(\mathcal{A}) \rangle$. A quasi-variety $\mathbf{Q}$ is *core-generated* if $\mathbf{Q} = \mathbb{ISPP}_U(\mathbf{K})$ for a class of core-generated algebras $\mathbf{K}$. An expanded algebra $\mathcal{A}$ is *equationally definable* by a finite set of equations Σ if $\mathsf{core}(\mathcal{A}) = \{x \in \mathcal{A} : \mathcal{A} \vDash \epsilon(x) \approx \delta(x) \text{ for all } \epsilon \approx \delta \in \Sigma\}$. A class of expanded algebras $\mathbf{K}$ is *(uniformly) equationally definable* if there is a finite set of equations Σ such that for all $\mathcal{A} \in \mathbf{K}$, $\mathcal{A}$ is equationally definable by Σ.

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Algebraizability of Weak Logics Let $\mathcal{F}m$ and Eq be respectively the set of formulas and equations in $\mathcal{L}$. We define two maps, $\tau : \mathcal{F}m \to \wp(\mathsf{Eq})$ and $\Delta : \mathsf{Eq} \to \wp(\mathcal{F}m)$ (also known as *transformers* [5]), that allow us to translate formulas into equations and vice versa. We say that τ and Δ are *structural* if for all substitutions $\sigma \in \mathsf{Subst}(\mathcal{L})$, $\tau \circ \sigma = \sigma \circ \tau$ and $\sigma \circ \Delta = \Delta \circ \sigma$. For any set of formulas Γ, we let $\tau(\Gamma) := \bigcup\{\tau(\phi) : \phi \in \Gamma\}$ and for all sets of equations Θ, we let $\Delta(\Theta) := \bigcup\{\Delta(\epsilon, \delta) : \epsilon \approx \delta \in \Theta\}$.

Definition 3 (Algebraizability)**.** A weak logic $\vdash$ is *algebraizable* if there are a core-generated quasivariety $\mathbf{Q}$, equationally definable by a finite set of equations Σ, a set of equations $\tau(x)$ and a set of formulas $\Delta(x, y)$ such that:

$$\Gamma \vdash \phi \iff \tau[\Gamma] \vDash^c_{\mathbf{Q}} \tau(\phi) \qquad\qquad \phi \dashv\vdash \Delta[\tau(\phi)]$$

$$\Delta[\Theta] \vdash \Delta(\eta, \delta) \iff \Theta \vDash^c_{\mathbf{Q}} \eta \approx \delta \qquad\qquad \eta \approx \delta \equiv^c_{\mathbf{Q}} \tau[\Delta(\eta, \delta)].$$

The quasivariety $\mathbf{Q}$ is then the *equivalent algebraic semantics* of the weak logic $\vdash$. As in the standard setting, the uniqueness of equivalent algebraic semantics holds.

Theorem 4. *If $(\mathbf{Q}_i, \Sigma_i, \tau_i, \Delta_i)_{i\in\{0,1\}}$ witness the algebraizability of $\vdash$, then:*

$$\mathbf{Q}_0 = \mathbf{Q}_1 \qquad \Sigma_0 \equiv_{\mathbf{Q}_0} \Sigma_1 \qquad \Delta_0(x, y) \dashv\vdash \Delta_1(x, y) \qquad \tau_0(\phi) \equiv^c_{\mathbf{Q}_0} \tau_1(\phi).$$

We apply the developed framework to the systems $\mathtt{InqB}$ and $\mathtt{InqB}^{\otimes}$ of classical propositional inquisitive and dependence logics to show that they are algebraizable. The intuitionistic versions $\mathtt{InqI}$ and $\mathtt{InqI}^{\otimes}$, however, are not — the core is the set of join-irreducible elements, which are not equationally definable.

Schematic Variants Following Ciardelli [4], we define the *schematic fragment* $\mathsf{Schm}(\vdash)$ of a weak logic $\vdash$ as $\mathsf{Schm}(\vdash) := \{(\Gamma, \phi) : \forall\sigma \in \mathsf{Subst}(\mathcal{L}),\ \sigma[\Gamma] \vdash \sigma(\phi)\}$. Then $\mathsf{Schm}(\vdash)$ is a standard logic and we write $\Gamma \vdash_S \phi$ if $(\Gamma, \phi) \in \mathsf{Schm}(\vdash)$.

Definition 5. A weak logic $\vdash$ is *finitely representable* if there is a set of formulas Λ such that for all Γ and ϕ : $\Gamma \vdash \phi \iff \Gamma \cup \{\sigma(\phi) : \phi \in \Lambda, \sigma \in \mathsf{At}(\mathcal{L})\} \vdash_S \phi$.

Finally, we can obtain a characterisation of algebraizable weak logics in terms of representability and algebraizability of the underlying schematic fragment.

Theorem 6. *A weak logic $\vdash$ is algebraizable iff $\mathsf{Schm}(\vdash)$ is algebraizable and $\vdash$ is finitely representable.*

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