Lifting of monotone-light factorizations

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Let $f\colon X \to Y$ be a continuous map between compact Hausdorff spaces. We say that f is *monotone*, respectively *light*, if for all $y \in Y$, the fiber $f^{-1}(y)$ is connected, respectively totally disconnected. These maps were studied by [Eil34] (for metric spaces) and [Why50], where they show that every continuous map f between compact Hausdorff spaces admits a factorization $f = g \circ h$ where g is light and h is monotone, unique up to a unique isomorphism. This is the so-called *monotone-light factorization* of compact Hausdorff spaces, which may be described as the pullback-stabilization and localization of the factorization system induced by the reflection $\pi_0\colon \mathsf{CHaus} \to \mathsf{Stn}$, a construction we make precise below, which maps each compact space X to its (Stone) space $\pi_0 X$ of connected components.

Suppose $\mathcal{C}$ has finite limits. In general, a *reflection* $R\colon \mathcal{C} \to \mathcal{D}$ (a functor with a fully faithful right adjoint) merely determines a prefactorization system $(\mathcal{L}, \mathcal{R})$ on $\mathcal{C}$. Here, $\mathcal{L}$ is class of morphisms f such that Rf is an isomorphism. Reflections for which $(\mathcal{L}, \mathcal{R})$ is a factorization system are said to be *simple*, as defined in [CHK85]. We note that the reflection $\mathsf{CHaus} \to \mathsf{Stn}$ is simple, with $\mathcal{L}$ the class of continuous maps which induce a homeomorphism on the underlying spaces of connected components.

This relationship between reflections and prefactorizations systems was extensively studied in [CHK85]. There, some properties of reflections are shown to imply simplicity. For example, *semi-left exact* reflections (also called *admissible* in the suitable context of Janelidze-Galois theory [BJ01]) are simple, as are reflections with *stable units*.

Given a factorization system $(\mathcal{L}, \mathcal{R})$, its pullback-stabilization and localization is a pair of classes of morphisms $(\mathcal{L}_{\mathsf{stab}}, \mathcal{R}_{\mathsf{loc}})$ defined by:

$$\begin{aligned} \mathcal{L}_{\mathsf{stab}} &= \{\, f \mid p^*(f) \in \mathcal{L} \text{ for all } p \,\}, \\ \mathcal{R}_{\mathsf{loc}} &= \{\, f \mid \text{there exists } p \text{ of effective descent such that } p^*(f) \in \mathcal{R} \,\}. \end{aligned}$$

It is not always the case that $(\mathcal{L}_{\mathsf{stab}}, \mathcal{R}_{\mathsf{loc}})$ is a factorization system; when it is, we say it is the *monotone-light factorization system* induced by $(\mathcal{L}, \mathcal{R})$.

The work of [CJKP97] was centered around studying conditions for which $(\mathcal{L}_{\mathsf{stab}}, \mathcal{R}_{\mathsf{loc}})$ is a factorization system. They found in 10.3 *ibid* that semi-left exactness is not sufficient to guarantee monotone-light factorizations, and further counter-examples were later given in [Xar04]. Nevertheless, Theorem 6.9 of [CJKP97] does characterize those factorization systems for which $(\mathcal{E}_{\mathsf{stab}}, \mathcal{M}_{\mathsf{loc}})$ is a factorization system, despite the conditions given therein being difficult to verify in general.

As part of a project aiming to study categorical Galois theory for various categorical structures, we study liftings of factorization systems and of simple, semi-left exact and stable units reflections, as well as ascertaining whether lifting pullback-stable/local classes preserves stability/locality. For example, suitable factorization systems for monoidal categories induce a factorization system for the categories of the respective enriched categories, and moreover, pullback-stability is preserved.

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For instance, consider the reflection $\mathsf{Cat} \to \mathsf{Ord}$, studied in [Xar03]. In Section 2.2 ibid it is shown that this reflection is simple, and the induced factorization system $(\mathcal{L}, \mathcal{R})$ admits a monotone-light factorization $(\mathcal{L}_{\mathsf{stab}}, \mathcal{R}_{\mathsf{loc}})$, both suitable in the aforementioned sense. The reflection lifts to a (simple) reflection $\mathsf{Cat}\text{-}\mathsf{Cat} \to \mathsf{Ord}\text{-}\mathsf{Cat}$, and the induced factorization system $(\overline{\mathcal{L}}, \overline{\mathcal{R}})$ is the lifting of $(\mathcal{L}, \mathcal{R})$. The main result of [Xar22] guarantees $\overline{\mathcal{R}_{\mathsf{loc}}} = \overline{\mathcal{R}}_{\mathsf{loc}}$, a non-trivial instance where a monotone-light factorization is lifted.

As another example, consider the monoidal reflection $R\colon \Delta \to [0,\infty]^{\mathrm{op}}$ of the quantale of distribution functions into the complete real half-line. This lifts to a left-exact reflection $\hat{R}\colon \Delta\text{-}\mathsf{Cat} \to [0,\infty]^{\mathrm{op}}\text{-}\mathsf{Cat}$ of probabilistic metric spaces (see [HR13]) into Lawvere metric spaces, which induces a stable, and therefore monotone-light, factorization system. This is lifted to the factorization system induced by $\hat{R}$, also monotone-light.

These lifting results are generally achieved in two steps: by expressing the various notions of factorization systems and reflections in 2-categories with reasonable properties, and by considering pseudofunctors which preserve certain bilimits between such 2-categories. Those pseudofunctors will also preserve those notions across 2-categories, allowing us to lift factorization systems and reflections from one context to another.

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