# On the concept of *Algebraic Crystallography*

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It is well known that, in the context of unital and strongly unital categories [1] and in the subtractive categories as well [7], [3], on any object $X$ there is at most one structure of abelian object. But, in these contexts, this did not seem so surprising because the three cases were closely related (strongly unital=unital+subtractive [5]) and because of the kind of their varietal origins: this uniqueness property arised naturally because, and when, some term in the definition of the varietal examples in question became a homomorphism in this variety.

Similar situations for other algebraic structures were even well known from a long time; for instance, it was clear that in a pointed Jónnson-Tarski variety, on any algebra $X$ there is at most one internal commutative monoid structure; the same property holds for the commutative and associative (=autonomous) Mal'tsev operations in the Mal'tsev varieties [6]. And again the limpid varietal contexts supplied the same simple explanation for this phenomenon.

But recently we were led to observe that the uniqueness structure for abelian objects still holds in the new context of Congruence hyperextensible categories [2]. This, in restrospect, emphasized that the uniqueness of the autonomous Mal'tsev operations was actually already noticed in Congruence Modular Varieties [4].

This phenomenon of uniqueness of some kinds of algebraic structures being now clearly extended to a much larger context than the one in the first paragraph, and the explanation by the existence of some kinds of terms in the definition of the varieties being no longer valid, it cannot remain possible to accept this uniqueness so easily and to keep it as an unquestioned process.

So, we propose to call *crystallographic for a given algebraic structure* any varietal or categorical setting in which, on any object $X$ of this setting, *there is at most one internal algebraic structure of this kind*, this terminology being chosen because, in such a setting, the algebraic structure in question is growing so scarce.

The aim of this talk will be to detail this situation in the context of Congruence hyperextensible varieties and categories, to establish the very first properties and general questionings about this *Algebraic Crystallography*, and finally to produce, from that, a spectacular observation with an example of an abelian category $Ab\mathbb{H}$ which i) fully faithfully embeds in a natural way the category $Ab$ of abelian groups and ii) in an independent way contains any category $K$-$Vect$ of $K$-vector spaces provided that the field $K$ is not of characteristic 2.

## References

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