Łukasiewicz logic properly displayed

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Mathematical fuzzy logics [4] are often motivated by semantic considerations, namely representing and reasoning about truth degrees. Hilbert systems are a convenient formalism for presenting logics corresponding to classes of algebras and they were abundantly used in presenting and organising various mathematical fuzzy logics, many of which come in large subclasses with specific properties [8].

Structural proof theory [15] studies the structure and properties of proofs and in this context sequent calculi are a fundamental tool in showing that proofs can be organised as to preserve *analyticity*. A core line of research (see for instance [14, 2, 3, 11, 10, 5]) focuses on the *algorithmic generation* of *analytic rules*, namely rules that preserve the analyticity whenever added to an analytic calculus.

Łukasiewicz logic is one of the most well-known and thoroughly studied mathematical fuzzy logics (see [13] for an overview of proof theoretic literature on mathematical fuzzy logics), and various (analytic) calculi capturing this logic were proposed and studied: for instance, [12] introduces various sequent-style calculi (hypersequent calculi, labelled sequent calculi and unlabelled sequent calculi) for the $\{\mathbf{0}, \rightarrow\}$-fragment of Łukasiewicz (and Abelian logic), while [1] introduces so-called relational hypersequent calculi for the full fragment of Łukasiewicz but $\{\mathbf{1}, \ominus\}$ (and Product and Gödel logics as well). Nonetheless, each calculus introduced in the literature so far exhibits some of the following features: non-standard readings of sequents[1] and non-standard introduction rules for logical operators (where the Łukasiewicz implication is a case in point).

The distinctive axiom of Łukasiewicz logic, namely $(A \rightarrow B) \rightarrow B \vdash A \vee B$,[2] is not *analytic-inductive* [10] (not even canonical) and it represents the main obstacle to a *uniform* and *modular* proof-theoretic treatment. Pivoting on an algebraic analysis of Łukasiewicz logic, we introduce a refinement of the general theory of display sequent calculi and algorithmic rule generation (as developed, for instance, in [6] and [10], respectively) aiming at surpassing this problem.

In particular, we rely on the fact that Łukasiewicz operators (see table below where the full language is considered) are not only normal operators, but also regular operators in the following sense (in [9] and [7] such operators are called 'double quasioperators'):

normal binary diamond	normal binary box
$A \odot \mathbf{0} = \mathbf{0} = \mathbf{0} \odot A$	$A \oplus \mathbf{1} = \mathbf{1} = \mathbf{1} \oplus A$
$(A \vee B) \odot C = (A \odot C) \vee (B \odot C)$	$(A \wedge B) \oplus C = (A \oplus C) \wedge (B \oplus C)$
$C \odot (A \vee B) = (C \odot A) \vee (C \odot A)$	$C \oplus (A \wedge B) = (C \oplus A) \wedge (C \oplus B)$
$A \ominus \mathbf{1} = \mathbf{0} = \mathbf{0} \ominus A$	$A \rightarrow \mathbf{1} = \mathbf{1} = \mathbf{0} \rightarrow A$
$(A \vee B) \ominus C = (A \ominus C) \vee (B \ominus C)$	$(A \vee B) \rightarrow C = (A \rightarrow C) \wedge (B \rightarrow C)$
$C \ominus (A \wedge B) = (C \ominus A) \vee (C \ominus B)$	$C \rightarrow (A \wedge B) = (C \rightarrow A) \wedge (C \rightarrow B)$
regular binary diamond	regular binary box
$(A \vee B) \oplus C = (A \oplus C) \vee (B \oplus C)$	$(A \wedge B) \odot C = (A \odot C) \wedge (B \odot C)$
$C \oplus (A \vee B) = (C \oplus A) \vee (C \oplus B)$	$C \odot (A \wedge B) = (C \odot A) \wedge (C \odot B)$
$(A \wedge B) \rightarrow C = (A \rightarrow C) \vee (B \rightarrow C)$	$(A \wedge B) \ominus C = (A \ominus C) \wedge (B \ominus C)$
$C \rightarrow (A \vee B) = (C \rightarrow A) \vee (C \rightarrow B)$	$C \ominus (A \vee B) = (C \ominus A) \wedge (C \ominus B)$

Exploiting the previous observation, we introduce a language expansion where the different "personalities" (normal versus regular) of the operators are fully-fledged and, in turn, it becomes possible

[1] E.g. the structural comma occurring in the antecedent and in the consequent of a sequent is interpreted as $\oplus$ in both cases, and the empty antecedent and the empty consequent of sequents is interpreted as $\mathbf{1}$ in both cases.

[2] Or any equivalent axiom in any complete fragment of Łukasiewicz logic.

introducing a sequent calculus with the so-called *relativized display property* (namely, every structure occurring in a derivable sequent is displayable). Moreover, all the logical introduction rules are standard and reflect the minimal order-theoretic properties of the operators, while the specific features of the logic are captured by so-called structural rules, so maintaining a neat division of labour that guarantees a modular treatment. At last, all the structural rules are automatically generated via (a specialisation of) the algorithm ALBA (to regular operators). Showing that the calculus enjoys (canonical) cut elimination is future work.

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