Modal reduction principles across relational semantics

Willem Conradie[3], Andrea De Domenico[1], Krishna Manoorkar[1], Alessandra Palmigiano[1,2], Mattia Panettiere[1*], Daira Pinto Prieto[4], and Apostolos Tzimoulis[1]

[1] Vrije Universiteit Amsterdam, the Netherlands
{m.panettiere,a.d.domenico,k.b.manoorkar,a.palmigiano,a.tzimoulis}@vu.nl
[2] University of Johannesburg, South Africa
[3] University of the Witwatersrand, South Africa
willem.conradie@wits.ac.za
[4] ILLC, University of Amsterdam, the Netherlands
d.pintoprieto@uva.nl

Previous work in the programme of unified correspondence [5, 4, 6] identified the classes of Inductive and Sahlqvist formulas for arbitrary logics that can be given algebraic semantics based on normal lattice expansions, viz. LE-logics. The members of these classes are characterised purely in terms of the order-theoretic properties of the algebraic interpretations of their connectives, and are unaffected by any change in the choice of particular semantics for the logic, as long as it is linked to the the algebraic semantics via a suitable duality. This leads to a modularization of the correspondence machinery whereby correspondents calculated uniformly by the ALBA calculus as conjunctions of set of pure quasi-inequalities. These can then be translated into first-order correspondents by applying the appropriate standard translation for the choice of dual relational semantics.

Here we approach the problem from the opposite end, by initiating a systematic comparison between the first-order correspondents of inductive formulas across different relational semantics. Some remarkable similarities between the first-order correspondents of certain well-known Sahlqvist axioms interpreted over different relational semantics have already been noted. For example, in [1] it was proven that the first-order correspondents of Sahlqvist formulas over Heyting algebra-valued Kripke frames are syntactically identical to their correspondents over ordinary Kripke frames, although the meaning is generalised (or shifted) as these formulas now belong to many-valued first-order predicate logic. In the setting of the epistemic logic of categories [2, 3] it was observed that, although formulas here denote categories rather than states of affairs, the epistemic meaning of standard axioms is arguably preserved and that moreover, the relational conditions they define (over, respectively, Kripke frames and polarity-based frames) resemble each other in very suggestive ways. For example, while the reflexivity condition defined by $p \to \Diamond p$ on Kripke frames can be expressed as $\Delta \subseteq R$ (where Δ denotes the identity relation), the same axiom imposed on polarity-based frames[1] the condition that $I \subseteq R$.

In the present work we build on these observations by building an environment in which it is possible to systematically compare the first-order correspondents of a given inductive formula across different relational semantics. Concretely, we restrict our attention to the Sahlqvist modal reduction principles (MRPs) [7] and focus on three relational settings, namely classical Kripke frames, polarity-based frames and many-valued polarity based frames. We will show that, if we write the first-order correspondents of Sahlqvist MRPs on Kripke frames in the right way, namely as inclusions of certain relational compositions, we can obtain their correspondents on polarity-based frames, roughly speaking, simply by reversing the direction of the inclusion

*Speaker.

[1]A polarity-based frame is a structure $(A, X, I, \mathcal{R})$ where A and X are non empty sets, $I \subseteq A \times X$ is the polarity relation and $\mathcal{R}$ is a family of additional relations compatible with I and used to interpret modalities.

and replacing everywhere (also in compositions of relations) the identity relation Δ with the polarity relation I. The correctness of this procedure turns on the fact that, just like the lifting from a Kripke frame to polarity-based frame preserves its complex algebra, it also "preserves" its associated relation algebra,[2] and so relational compositions and pseudo-compositions on Kripke frames can be systematically lifted to I-mediated and non I-mediated compositions of relations on polarity-based frames. The relations Δ and I thus play the role of parameters in the correspondence. This parametricity phenomenon was already observed when moving from *crisp* polarity-based frames to *many-valued* polarity-based frames. Here the relevant parameter is the truth-value algebra, which changes from the Boolean algebra $\mathbf{2}$ to an arbitrary complete Heyting algebra $\mathbf{A}$ while, syntactically, the first-order correspondents of Sahlqvist MRPs remain verbatim the same. This latter result partially generalizes that of [1] to the polarity-based setting, and provides an analogous result lifting correspondence along the dashed arrow in the following commutative diagram:

$$\begin{tikzcd}
\text{Polarity-based frames} \arrow[r] & \text{MV Polarity-based frames} \\
\text{Kripke frames} \arrow[u, hook] \arrow[r] & \text{MV Kripke frames} \arrow[u, hook, dashed]
\end{tikzcd}$$

The results presented here do not generalize smoothly beyond MRPs, as there are Sahlqvist axioms whose correspondents over polarity-based frames are not equivalent to the liftings of their correspondents on Kripke frames, e.g. $\Diamond(p \vee q) \leq \Diamond(p \wedge q)$. We conjecture that this failure is due to the loss of distributivity when moving from classical modal logic to general LE-logics and that accordingly, for general LE-logics, the present result can be generalized to all inductive inequalities which do not contain $\wedge$ of $\vee$.

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[2] By which we mean the relation algebra with constants corresponding to the identity relation and the accessibility relation together with various notions of composition on the side of Kripke frames, while on the polarity-based frame side we have constants for the incidence relation as well as the additional relations together with suitable notions of composition.